Generated Necessities and Possibilities

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A necessity measure is a function n from a Boolean algebra B to the real interval [0,1] such that $n(x \wedge y) = \min\{n(x), n(y)\}$ for every $x, y \in B$ and $n(0) = 0$, $n(1) = 1$. Necessities are strictly related to Shafer's consonant belief functions and are basic tools when dealing with imprecision and uncertainty. In this article we propose a technique to generate necessities given a collection of items of information quantified by an initial valuation. The method we employ enables us to define conditional necessities in a very natural way and the composition of two necessities by a rule analogous to Dempster's rule. This is obtained by skipping the condition $n(0) = 0$ and therefore considering necessities with a nonzero "degree of inconsistency."

I. INTRODUCTION

In this article we examine some mathematical features of possibility and necessity measures that, as D. Dubois and H. Prade point out,[1] are basic tools to represent imprecision and to quantify uncertainty. The main result is a technique we propose to generate a necessity (or a possibility) given an initial valuation of the events (see Section III): this technique is more general and substantially different from Shafer's building of consonant belief functions upon a probability mass distribution in that the former is of lattice theoretical kind, while the latter employs the additive structure of the real number interval [0,1]. Because of its lattice nature, our technique could be applied even to necessities and possibilities which eventually took values in a complete lattice.

The formulas we obtain for the necessities and possibilities generated by an initial valuation enable us to define conditional necessities and possibilities. The notion we arrive at is very close to what D. Dubois and H. Prade have proposed.[1]

It is worth noting that we do not require the null event to be valued zero; so, the arising class of necessities (and possibilities as well) turn out a complete lattice, the maximum being the map constantly equal to one. In particular, the

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join of two necessities is defined also in the case of conflicting necessities and determines a rule analogous to Dempster's rule of composition.

II. PRELIMINARIES

In the sequel B denotes a Boolean algebra whose elements are called *events*, we denote by 0 and 1 the minimum and the maximum, respectively. The class $F(B)$ of the maps from B to $[0,1]$ is a complete lattice with respect to the infinitary operations $\wedge$ and $\vee$ defined by

$$(\wedge s_i)(x) = \inf \{s_i(x)/i \in I\};\ (\vee s_i)(x) = \sup \{s_i(x)/i \in I\}$$

where $\langle s_i \rangle_{i \in I}$ is any family of elements of $F(B)$. If $\alpha \in [0,1]$, then the subset $C(s,\alpha) = \{x \in B/s(x) \geq \alpha\}$ is called the *α-cut* of s. We say that an element n of $F(B)$ is a *necessity* if

$$n(x \wedge y) = n(x) \wedge n(y) \qquad \text{and} \qquad n(1) = 1, \tag{2.1}$$

and we denote by $N(B)$ the set of the necessities defined on B. We call *degree of inconsistency* the number $n(0)$ and we denote it by $I(n)$; since n is increasing, $I(n)$ is the minimum of n. We say that n is *completely consistent* if $I(n) = 0$ and that n is *completely inconsistent* if $I(n) = 1$. Only the values of n greater than $I(n)$ are meaningful and $n(x) = I(n)$ corresponds to lack of information about x. In particular, the completely inconsistent necessity gives no information about the events.

We say that an element p of $F(B)$ is a *possibility* if

$$p(x \vee y) = p(x) \vee p(y) \qquad \text{and} \qquad p(0) = 0. \tag{2.2}$$

We denote by $P(B)$ the class of the possibilities defined on B; we call *degree of consistence* the maximum $p(1)$, and denote it by $C(p)$. Moreover, we say that p is *completely consistent* if $C(p) = 1$ and *completely inconsistent* if $C(p) = 0$. Given an element f of $F(B)$, define $\sim f$ by $\sim f(x) = 1 - f(-x)$. The operation $\sim$ fulfills the following properties

$$\sim(\sim f) = f, f \leq g \Rightarrow \sim f \geq \sim g, \sim(\vee f_i) = \wedge(\sim f_i), \sim(\wedge f_i) = \vee(\sim f_i) \tag{2.3}$$

where $f, g \in F(B)$ and $\langle f_i \rangle$ is any family of elements of $F(B)$. Now, it is immediate to prove that

$$f \text{ necessity} \Leftrightarrow \sim f \text{ possibility}$$

so the concept of possibility is dual of the concept of necessity. As a consequence in this article we sometimes limit ourselves to examine the necessities.

The following propositions are obvious generalizations of well-known results.[1]

PROPOSITION 2.1. *If n is a necessity then, for every $x, y \in B$,*
(a) *either* $n(x) = I(n)$ or $n(-x) = I(n)$;
(b) $n(x \vee y) \geq n(x) + n(y) - n(x \wedge y)$.

Proof. If n is a necessity, $n(x) \wedge n(-x) = n(x \wedge -x) = n(0) = I(n)$ and this proves (a). Inequality (b) is obvious.

Recall that a *filter* of B is a subset F of B such that $x, y \in F \Rightarrow x \wedge y \in F$; $x \in F$ and $y \geq x \Rightarrow y \in F$. F is *proper* if $F \neq B$, F is *principal* if there exists $e \in B$ such that $F = \{x \in B/x \geq e\}$. The following proposition gives some obvious characterizations of the necessities.

PROPOSITION 2.2. *Let n be an element of $F(B)$ such that $n(1) = 1$, then the following propositions are equivalent:*
(a) *n is a necessity;*
(b) *n is increasing and $n(x \wedge y) \geq n(x) \wedge n(y)$;*
(c) *$n(x \wedge y) \geq n(x) \wedge n(y)$ and $n(x \vee y) \geq n(x) \vee n(y)$;*
(d) *n is closed with respect to Modus Ponens, that is, $n(y) \geq n(x \to y) \wedge n(x)$.*

Proof. That (a), (b), and (c) are equivalent is obvious.
(a) $\Rightarrow$ (d) If n is a necessity, then $n(y) \geq n(x \wedge y) = n((x \to y) \wedge x) = n(x \to y) \wedge n(x)$.
(d) $\Rightarrow$ (a) We have that $n(y) \geq n((x \wedge y) \to y) \wedge n(x \wedge y) = n(1) \wedge n(x \wedge y) = n(x \wedge y)$. Likewise, $n(x) \geq n(x \wedge y)$ and therefore $n(x) \wedge n(y) \geq n(x \wedge y)$. In particular, this entails that n is increasing. On the other hand, it is also $n(x \wedge y) \geq n(x \to (x \wedge y)) \wedge n(x) = n(-x \vee (x \wedge y)) \wedge n(x) = n(-x \vee y) \wedge n(x) \geq n(y) \wedge n(x)$.

The following proposition relates the necessities with the filters; the links between necessity measures and filters where first pointed out by U. Hohle.[2]

PROPOSITION 2.3. *A map $n{:}B \to [0,1]$ is a necessity if and only if every cut of n is a filter of B. Moreover, the necessities can be identified with the families $\langle C_\alpha \rangle_{\alpha \in I}$ of filters of B with I complete subset of $(0,1]$ and*

$$\cap C_{\alpha_i} = C_{\vee \alpha_i}. \tag{2.4}$$

for every family $\langle \alpha_i \rangle$ *of elements of I.*

Proof. It is immediate that every cut of a necessity is a filter. Conversely, assume that $C(n, \alpha)$ is a filter for every $\alpha \in [0, 1]$ and let $\alpha = n(x) \wedge n(y)$. Then, since x and y belong to $C(n,\alpha)$, $C(n, \alpha)$ contains also $x \wedge y$ and therefore $n(x \wedge y) \geq n(x) \wedge n(y)$. To prove that n is increasing, let $x \geq y$ and $\alpha = n(y)$. Since y belongs to $C(n, \alpha)$, also x belongs to $C(n, \alpha)$ and we have $n(x) \geq \alpha = n(y)$; this proves that n is a necessity.

We associate to every necessity n the family $\langle C(n, \alpha)\rangle_{\alpha \in (0,1]}$ of filters; then it is easy to prove that (2.4) is satisfied. Conversely, let $\langle C_\alpha \rangle_{\alpha \in I}$ be a family of filters of B satisfying (2.4) and define the map $n{:}B \to [0, 1]$ by

$$n(x) = \begin{cases} \vee\{\beta \in I/x \in C_\beta\} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases} \tag{2.5}$$

We have that, for every $\alpha \in I$, $C(n, \alpha) = C_\alpha$; indeed, if $x \in C_\alpha$ then $n(x) \geq \alpha$, that is $C_\alpha \subseteq C(n, \alpha)$. If $x \in C(n, \alpha)$ and $x \neq 1$ then $\bigvee\{\beta \in I/ x \in C_\beta\} \geq \alpha$. Since (2.4) implies that the family $\langle C_\alpha \rangle_{\alpha \in I}$ is decreasing, we have $C_{\bigvee\{\beta \in I/x \in C\}} \subseteq C_\alpha$ and, by (2.4), $\cap\{C_\beta/x \in C_\beta\} \subseteq C_\alpha$. Thus $x \in C_\alpha$ and $C(n, \alpha) \subseteq C_\alpha$.

If $\alpha \notin I$ and $\{\beta \in I/\beta \geq \alpha\}$ is nonempty, then $C(n, \alpha) = C(n, \alpha') = C_{\alpha'}$ where $\alpha' = \bigwedge\{\beta \in I/\beta \geq \alpha\}$. If $\{\beta \in I/\beta \geq \alpha\}$ is empty then $C(n, \alpha) = \{1\}$. Thus the cuts of n are filters and therefore n is a necessity.

As an example, any finite chain of filters

$$F_{\alpha_1} \supset \ldots \supset F_{\alpha_m} \qquad \text{with} \qquad 0 \leq \alpha_1 \leq \ldots \leq \alpha_m \leq 1$$

defines a necessity; in particular, the (characteristic functions of the) filters of B are necessities. Since the filters in a Lindenbaum Boolean algebra coincide with the theories, the necessities can be viewed as a generalization of the notion of a theory in the first order logic. This suggests the following considerations. Let T be a theory and α a formula, then $\alpha \notin T$ does not mean that α is false but, in a sense, that we have not sufficient informations in order to prove α. The theory T expresses the falsity of α only if the negation $-\alpha$ of α belongs to T. Analogously, if n is a necessity, then $n(\alpha) = 0$ [more generally $n(\alpha) = I(n)$] does not mean that, in our opinion, α is false but that we have not enough information in order to support our belief in α. As a matter of fact, informations about the falsity of α are given by $n(-\alpha)$.

III. GENERATED NECESSITIES AND POSSIBILITIES

In the sequel an *initial valuation* is any map f from B into [0,1]. The elements of the support $D_f = \{x \in B/f(x) \neq 0\}$ are called *focal events* of f. In this section we examine the question of generating a necessity and a possibility in accordance with an initial valuation.

PROPOSITION 2.1. *The meet $\bigwedge n_i$ of a family $\langle n_i \rangle$ of necessities is a necessity. If f is an initial valuation then the necessity $\bar{f} = \bigwedge\{g \in N(B)/g \geq f\}$ can be obtained by*

$$\bar{f}(z) = \begin{cases} 1 & \text{if } z = 1 \\ \bigvee\{f(y_1) \wedge \ldots \wedge f(y_m)/y_1 \wedge \ldots \wedge y_m \leq z\} & \text{if } z \neq 1. \end{cases} \tag{3.1}$$

Proof. To prove that $\bar{f}$ is a necessity, let z_1, z_2 be elements in $B - \{1\}$, then $\bar{f}(z_1) \wedge \bar{f}(z_2) = [\bigvee\{f(y_1) \wedge \ldots \wedge f(y_n)/y_1 \wedge \ldots \wedge y_n \leq z_1\}] \wedge [\bigvee\{f(w_1) \wedge \ldots \wedge f(w_m)/w_1 \wedge \ldots \wedge w_m \leq z_2\}] = \bigvee\{f(y_1) \wedge \ldots \wedge f(y_n) \wedge f(w_1) \wedge \ldots \wedge f(w_m)/y_1 \wedge \ldots \wedge y_n \leq z_1$ and $w_1 \wedge \ldots \wedge w_m \leq z_2\} \leq \bigvee\{f(u_1) \wedge \ldots \wedge f(u_s)/u_1 \wedge \ldots \wedge u_s \leq z_1 \wedge z_2\} = \bar{f}(z_1 \wedge z_2)$. Since $\bar{f}(1) = 1$ and $\bar{f}$ is increasing, by (b) of Proposition 2.2 $\bar{f}$ is a necessity. Moreover, if n is a necessity and $n \geq f$ then, for every $y_1, \ldots y_n$ such that $y_1 \wedge \ldots \wedge y_n \leq z$, we have $n(z) \geq n(y_1 \wedge$

$\ldots \wedge y_n) = n(y_1) \wedge \ldots \wedge n(y_n) \geq f(y_1) \wedge \ldots \wedge f(y_n)$ and therefore $n(z) \geq \bar{f}(z)$. So $\bar{f}$ is the smallest necessity greater than f.

We say that $\bar{f}$ is the necessity *generated* by the initial valuation f: in a sense $\bar{f}$ can be viewed as the "theory" generated by the "system of axioms" f. Notice that in the class of completely consistent necessities the operator $^-$ is not always defined; indeed it is possible that no completely consistent necessity is greater than f. This is the main reason for which we have skipped the condition $n(0) = 0$ in defining the necessities. We call *degree of inconsistency* $I(f)$ of f the degree of inconsistency $I(\bar{f})$ of $\bar{f}$, and, obviously,

$$I(f) = \bigvee\{f(y_1) \wedge \ldots \wedge f(y_n)/y_1 \wedge \ldots \wedge y_n = 0\}. \tag{3.2}$$

Notice that the events x such that $f(x) = 0$ have no influence in determining the necessity $\bar{f}$. This is in accordance with the fact that $f(x) = 0$ means that we have no opinion about the event x; not that we think that x is false. Obviously, also the events x for which $f(x) \leq I(f)$ give no useful informations in the construction of $\bar{f}$. Indeed, the initial valuation f carries on an amount of information but it is possible that some informations are inconsistent. The number $I(f)$ computed by (3.2) measures this inconsistency and indicates what values of f are meaningful in the construction of $\bar{f}$.

It is natural to assume that the initial valuation f is finite, that is, it is addressed only to a finite number of focal events, $D_f = \{e_1, \ldots, e_n\}$. In this case (3.1) defines $\bar{f}$ in a constructive simple way, obviously.

The following are simple examples.

Example 3.1. (Only an element is focal). Assume $D_f = \{e\}$ and $f(e) = \alpha$, then f generates the necessity n^e_α defined by

$$n^e_\alpha(x) = \begin{cases} 1 & \text{if } x = 1; \\ \alpha & \text{if } x \geq e \\ 0 & \text{otherwise.} \end{cases}$$

In the case $\alpha = 1$, n^e_1 is the characteristic function of the principal filter generated by e and will be denoted by n^e.

Example 3.2. (There are only two focal events). Let $D_f = \{e_1, e_2\}$, $f(e_1) = 1/2$ and $f(e_2) = 1$. Then

$$\bar{f}(x) = \begin{cases} 1 & \text{if } x \geq e_2 \\ 0 & \text{if } x \ngeq e_1 \wedge e_2 \\ 1/2 & \text{if } x \geq e_1 \wedge e_2 \text{ and } x \ngeq e_2 \end{cases}$$

and $I(f) = 1/2$ if $e_1 \wedge e_2 = 0$ while $I(f) = 0$ otherwise.

We say that an initial valuation is *consonant* if D_f is a finite chain $e_1 < e_2 < \ldots < e_k$, and $f(e_1) \leq f(e_2) \leq \ldots \leq f(e_k) = 1$. In this case $\bar{f}(x) = \sup \{f(e_i)/e_i \leq x\}$ and, if $e_1 \neq 0$, $\bar{f}$ is completely consistent.

PROPOSITION 3.2. *For every necessity n the following are equivalent:*

(a) *n is generated by a finite initial valuation;*

(b) *n is generated by a consonant initial valuation.*

Proof. (a) $\Rightarrow$ (b) Let n be generated by the finite initial valuation f, and set $n(B) = \{\alpha_1, \ldots, \alpha_j\}$ with $\alpha_j = 1 > \ldots > \alpha_1$. We claim that every cut $C(n,\alpha_i)$ is a principal filter generated by the meet $e_i = \wedge\{x \in B/f(x) \geq \alpha_i\}$. Indeed, $n(x) \geq \alpha_i \Rightarrow \exists a_1 \ldots \exists a_k \in B$, $a_1 \wedge \ldots \wedge a_k \leq x, f(a_1) \geq \alpha_i, \ldots, f(a_k) \geq \alpha_i \Rightarrow x \geq e_i$. Conversely, since there exist $a_1, \ldots, a_k \in D_f$ such that $f(a_1) \geq \alpha_i, \ldots, f(a_k) \geq \alpha_i$ and $e_i = a_1 \wedge \ldots \wedge a_k$, we have that $n(e_i) \geq \alpha_i$ and therefore $x \geq e_i \Rightarrow n(x) \geq n(e_i) \geq \alpha_i$. We have $e_i < e_{i+1}$ because $C(n,\alpha_{i+1})$ is strictly contained in $C(n,\alpha_i)$. On the other hand, for every $x \in B$ there exists $i \in \{1, \ldots, j\}$ such that $n(x) = \alpha_i = n(e_i)$ and this proves that $n(x) = \vee\{n(e_j)/e_j \leq x\}$. Thus n is a necessity generated by a consonant initial valuation, namely the restriction of n to $\{e_1, \ldots, e_j\}$.

The implication (b) $\Rightarrow$ (a) is immediate.

In particular, if B is a finite Boolean algebra, then every necessity can be generated by a consonant valuation while, if B is infinite, then this is not true. Indeed, in this case a nonprincipal filter exists in B and its characteristic function is a necessity that is not generated by a finite valuation (otherwise, as observed in proving Proposition 3.2, its cuts should be all principal).

The dual of Proposition 3.1 holds.

PROPOSITION 3.3. *The join $\vee p_i$ of a family $\langle p_i \rangle$ of possibilities is a possibility. In particular, if f is an initial valuation then $\mathring{f} = \vee\{g \in P(B)/g \leq f\}$ is a possibility and $\mathring{f}$ can be obtained by*

$$\mathring{f}(z) = \begin{cases} 0 & \text{if} \quad z = 0 \\ \wedge\{f(y_1) \vee \ldots \vee f(y_n)/y_1 \vee \ldots \vee y_n \geq z\} & \text{if} \quad z \neq 0. \end{cases} \tag{3.3}$$

We say that $\mathring{f}$ is the *possibility generated by* f, we set $C(f) = C(\mathring{f})$ and we say that $C(f)$ is the *degree of consistence* of f. Obviously, initial valuations should be taken keeping in mind if we want to evaluate the possibility or the necessity of the events under consideration. For instance, while generating the necessity measure we consider only the values of f that are different from zero, if we are interested in generating the possibility measure the events x such that $f(x) = 1$ have no influence. Indeed, $f(x) = 1$ expresses lack of information about the event x in that we do not know reasons to disbelieve in x.

In particular, if B is finite, then $\mathring{f}$ is completely determined by the associated possibility distribution, that is its restriction π to the atoms of B. Now, if z is an atom (3.3) gives $\pi(z) = \wedge_{y \geq z} f(y)$, so we have the following very simple formula

$$\mathring{f}(x) = \vee\{\pi(z)/z \text{ atom } z \leq x\} = \vee\{\wedge_{y \geq z} f(y)/z \text{ atom}, z \leq x\}.$$

PROPOSITION 3.4. *The necessity $\bar{f}$ is completely consistent if and only if D_f satisfies the "finite intersection property" that is every finite meet of elements of D_f is different from 0.*

Proof. $I(f) = 0 \Leftrightarrow \bigvee\{f(x_1) \wedge \ldots \wedge f(x_n)/x_1 \wedge \ldots \wedge x_n = 0\} = 0 \Leftrightarrow \forall x_1, \ldots, x_n \in B$ such that $x_1 \wedge \ldots \wedge x_n = 0$ an index i exists such that $f(x_i) = 0 \Leftrightarrow \forall x_1, \ldots, x_n \in D_f$ we have $x_1 \wedge \ldots \wedge x_n \neq 0$.

The following proposition shows that the operators $^-$ and $^\circ$ have properties very like to the closure and interior operators in topology theory.

PROPOSITION 3.5. *The following hold:*

$$f \text{ necessity} \Leftrightarrow f = \bar{f}; \tag{3.4}$$

$$f \text{ possibility} \Leftrightarrow f = \mathring{f}; \tag{3.5}$$

$$\mathring{f}(x) \leq f(x) \leq \bar{f}(x) \qquad \text{for every } x \in B; \tag{3.6}$$

$$\sim\bar{f} = (\sim f)^\circ; \sim\mathring{f} = (\sim f)^-; (\bar{f})^- = \bar{f}; (\mathring{f})^\circ = \mathring{f}; \tag{3.7}$$

$$f \leq g \Rightarrow \bar{f} \leq \bar{g}; f \leq g \Rightarrow \mathring{f} \leq \mathring{g}. \tag{3.8}$$

Proof. We limit ourselves to prove the first equality of (3.7). Indeed, if $z \neq 0$, $\sim(\bar{f})(z) = 1 - \bar{f}(-z) = 1 - \bigvee\{f(x_1) \wedge \ldots \wedge f(x_n)/x_1 \wedge \ldots \wedge x_n \leq -z\} = \bigwedge\{(1 - f(x_1)) \vee \ldots \vee (1 - f(x_n))/-x_1 \vee \ldots \vee -x_n \geq z\} = \bigwedge\{(\sim f)(y_1) \vee \ldots \vee (\sim f)(y_n)/y_1 \vee \ldots \vee y_n \geq z\} = (\sim f)^\circ(z)$.

An initial valuation $f: B \to [0,1]$ is called *n-stable* if $\bar{f}(x) = f(x)$ for every $x \in D_f$. The following is an immediate consequence of Proposition 3.1 (see also Zhang Guangquan[3]).

PROPOSITION 3.6. *An initial valuation f with $f(1) = 1$ is n-stable if and only if, for every $x_1, \ldots, x_n \in D_f$ and $z \in D_f$*

$$x_1 \wedge \ldots \wedge x_n \leq z \Rightarrow f(x_1) \wedge \ldots \wedge f(x_n) \leq f(z).$$

A simple consequence of Proposition 3.6 is that if D_f is a chain, then f is n-stable if and only if it is increasing.

IV. THE LATTICES OF THE NECESSITIES AND THE POSSIBILITIES

Proposition 3.1 shows that $N(B)$ is a complete lattice and that the meets in $N(B)$ coincide with the meets in $F(B)$. The maximum of $N(B)$ is the map constantly equal to 1, the minimum is the map n such that $n(1) = 1$ and $n(x) = 0$ if $x \neq 1$; in $N(B)$ there is no atom or co-atom. $N(B)$ is not a sublattice of $F(B)$ since given a family $\langle n_i \rangle_{i \in I}$ of necessities, the join in $N(B)$ is equal to $(\bigvee n_i)^-$ and it is, in general, different from the join $\bigvee n_i$ in $F(B)$. Let n_1 and n_2 be two necessities, we call *sum* $n_1 + n_2$ of n_1 and n_2 its join $(n_1 \vee n_2)^-$ in $N(B)$. If the condition

$$\forall x \forall y \forall i \forall j \exists h\, n_i(x) \wedge n_j(y) \leq n_h(x) \wedge n_h(y) \tag{4.1}$$

is satisfied, then it is easily proven that the joins in $F(B)$ coincide with the joins in $N(B)$. In particular this happens for directed families of necessities.

Likewise, Proposition 3.4 shows that the class $P(B)$ of the possibilities is a complete lattice and that the joins in $P(B)$ coincide with the joins in $F(B)$. The meet in $P(B)$ of a family $\langle p_i \rangle$ of possibilities is the possibility $(\bigwedge p_i)^\circ$. If p_1 and p_2 are two possibilities, we define the *product* $p_1 \cdot p_2$ as the meet $(p_1 \wedge p_2)^\circ$ in $P(B)$.

PROPOSITION 4.1. *If n_1 and n_2 are two necessities and p_1 and p_2 are two possibilities then, for every $z \in B$*

$$\sim(n_1 + n_2) = (\sim n_1) \cdot (\sim n_2); \sim(p_1 \cdot p_2) = (\sim p_1) + (\sim p_2), \tag{4.2}$$

so, the map $\sim$ is a dual isomorphism between the lattices $N(B)$ and $P(B)$.

Proof. Obvious.

PROPOSITION 4.2. *If n_1 and n_2 are two necessities and p_2 and p_2 are two possibilities then, for every $z \in B$*

$$\begin{aligned}(n_1 + n_2)(z) &= \bigvee\{n_1(x) \wedge n_2(y)/x \wedge y \leq z\};\\ (p_1 \cdot p_2)(z) &= \bigwedge\{p_1(x) \vee p_2(y)/x \vee y \geq z\}\end{aligned} \tag{4.3}$$

$$\begin{aligned}I(n_1 + n_2) &= \bigvee\{n_1(x) \wedge n_2(-x)/x \in B\};\\ C(p_1 \cdot p_2) &= \bigwedge\{n_1(x) \vee n_2(-x)/x \in B\}.\end{aligned} \tag{4.4}$$

Proof. We have $(\bigwedge n_1 \vee n_2)^-(z) = \bigvee\{(n_1 \vee n_2)(x_1) \wedge \ldots \wedge (n_1 \vee n_2)(x_m)/ x_1 \wedge \ldots \wedge x_m \leq z\} = \bigvee\{n_{i_1}(x_1) \wedge \ldots \wedge n_{i_m}(x_m)/x_1 \wedge \ldots \wedge x_m \leq z \text{ and } i_1, \ldots, i_m \in \{0,1\}\} = \bigvee\{n_1(x) \wedge n_2(y)/x \wedge y \leq z\}$.

The remaining part of the proposition is obvious.

We point out that if f_1 and f_2 are valuations then $(f_1 \vee f_2)^- = \bar{f}_1 + \bar{f}_2$ and $(f_1 \wedge f_2)^- = \bar{f}_1 \cdot \bar{f}_2$. As a consequence, if f is a finite initial valuation then it is possible to decompose $\bar{f}$ as follows.

$$\bar{f} = n_{\alpha_1}^{e_1} + \ldots + n_{\alpha_m}^{e_m} \tag{4.5}$$

where $D_f = \{e_1, \ldots, e_m\}$, $f(e_i) = \alpha_i$, and n_α^e is defined as in Example 3.1.

In the case B finite the second formula of (4.3) enables us to write the possibility distribution π_{12} of the product $p_1 \cdot p_2$ in a very simple way. Indeed if z is an atom, since $p_1(x) \vee p_2(y) \geq p_1(z) \wedge p_2(z)$ for every x, y such that $x \vee y \geq z$, we have

$$\pi_{12}(z) = \pi_1(z) \wedge \pi_2(z),$$

where π_1 and π_2 are the distributions of p_1 and p_2, respectively.

The number $I(n_1 + n_2)$ given by (4.4) represents a measure of the conflict between two necessities n_1 and n_2. If the conflict of opinion about an event x is complete, that is $n_1(x) = 1$ and $n_2(-x) = 1$, then $I(n_1 + n_2) = 1$ and $n_1 + n_2$ is completely inconsistent.

V. CONDITIONAL NECESSITIES

Let n be a necessity, $a \in B$ an event and consider the necessity n^a defined in Example 3.1, that is $n^a(x) = 1$ if $x \geq a$, $n^a(x) = 0$ otherwise. We call *conditional necessity given the event a* the necessity $n + n^a$ and we set

$$n(x/a) = (n + n^a)(x). \tag{5.1}$$

In other words the conditional necessity given the event a is obtained by adding to n the information "a is true."

PROPOSITION 5.1. *Given a necessity n and an event a,*

(i) $n(x/a) = n(a \to x)$; $n(a/a) = 1$, $n(x/1) = n(x)$;
(ii) $n(x \wedge y/a) = n(x/a) \wedge n(y/a)$; $n(x/a \vee b) = n(x/a) \wedge n(x/b)$;
(iii) $n(a) \wedge n(x/a) = n(x \wedge a)$; $n(x) = n(x/a) \wedge n(x/-a)$;
(iv) $n(a) \wedge n(x/a) = n(x) \wedge n(a/x)$; $n(x/a/b) = n(x/a \wedge b)$.

Proof. In order to prove the first equality in (i), since n is increasing and $-a \vee z$ is the maximum of the set $\{x \in B/x \wedge a \leq z\}$, we have

$$\begin{aligned}(n + n^a)(z) &= \bigvee\{n(x) \wedge n^a(y)/x \wedge y \leq z\} = \bigvee\{n(x)/x \wedge y \leq z \text{ and } y \geq a\} \\ &= \bigvee\{n(-y \vee z)/y \geq a\} = n(-a \vee z).\end{aligned}$$

The second equality in (ii) follows from

$$\begin{aligned}n(x/a \vee b) = n(a \vee b \to x) = n((-a \wedge -b) \vee x) &= n((-a \vee x) \wedge (-b \vee x)) \\ &= n(a \to x) \wedge n(b \to x).\end{aligned}$$

The remaining part of the proposition is obvious.

The first equality in (i) shows the link between our definition of conditional necessity and the notion given by D. Dubois and H. Prade (Ref. 1, Chap. 4). Namely they proposed

$$n(x/a) = \begin{cases} n(a \to x) & \text{if} \quad n(-a) < n(a \to x) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $n(-a) = 0$, the two conditional necessities coincide; but in general the conditional necessity we define has degree of inconsistency $I(n(-/a))$ equal to $n(-a)$. This means that in the case $n(-a) > I(n)$ the additional information "a is true" originates a conflict with the previous belief. This conflict requires to raise the threshold of validity of the necessity degrees. Observe that $n(-a/a) = n(-a)$: this is not surprising in our approach since only values greater that $n(-a)$ can be interpreted as effective truth assertions.

Equality (iv) is analogous to Bayes formula but unfortunately it is not possible to obtain $n(a/x)$ as a function of $n(x/a)$, $n(x)$ and $n(a)$ as in the probabilistic case. Indeed, assume that h is such a function, that is, $n(x/a) = h(n(a), n(x), n(a/x))$ for every pair of events x, a. In particular, if n is the necessity generated by an event $e \neq 0$, then

$$n^{e\wedge a}(x) = h(n^e(a), n^e(x), n^{e\wedge x}(a)).$$

Now, if e, a, and x satisfy $e \wedge x \nleq a$, $a \leq x$, $e \nleq x$, then, since $e \nleq a$ and $e \wedge a \leq x$, we have $h(0,0,0) = 1$. On the other hand, if $e \wedge a \nleq x$ and $e \wedge x \nleq a$, then, since $e \nleq a$, $e \nleq x$, we have $h(0,0,0) = 0$, an absurdity.

VI. CONCLUDING REMARKS

We summarize some features of the proposed approach.

First of all it furnishes a uniform treatment of the composition of necessities and the conditioning necessities. This treatment is achieved by the very elegant and simple tools of the lattice theory.

Moreover, as a consequence of the implication $f_1 \leq f_2 \Rightarrow \bar{f}_1 \leq \bar{f}_2$, we have the monotonicity of the composition and conditioning. This is not true for the analogous concepts known in literature but it is in accordance with the idea that a measure of necessity summarizes a corpus of imprecise informations and therefore it increases when the informations increase.

Finally, the degree of inconsistency we have introduced seems to be a useful control tool to dominate conflicting informations.

References

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