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GEOMETRY WITHOUT POINTS

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A number of authors have addressed "the well-known conviction that geometry may be built without points" ([5]; see also the other references). In this paper we propose an approach to metric spaces, and thence to geometry, based on the primitives: solid, inclusion, and distance between solids.

DEFINITION. A *space of solids* is a set S of elements called *solids*, together with a partial order relation $\leqslant$ on S, called *inclusion*, and a function δ from $S \times S$ to the nonnegative real numbers, called *distance*, such that, for all solids u, v, w:

(1) if $u \leqslant w$, then $\delta(u,v) \geqslant \delta(v,w)$,

(2) the *diameter* $\Delta(v) = \sup\{\delta(p,g) : p \leqslant v \text{ and } g \leqslant v\}$ is finite, and

$$\delta(u,v) + \delta(v,w) + \Delta(v) \geqslant \delta(u,w),$$

(3) $u \leqslant v$ if and only if $\delta(u',v) = 0$ for every u' included in u,

(4) if $\varepsilon > 0$, then there are solids u', v' such that

$$u' \leqslant u, \quad v' \leqslant v, \quad \Delta(u') \leqslant \varepsilon, \quad \Delta(v') \leqslant \varepsilon, \quad \text{and} \quad \delta(u',v') = \delta(u,v).$$

A *point* in a space of solids $(S, \leqslant, \delta)$ is a minimal element of $(S, \leqslant)$. An *ε-point*, where $\varepsilon > 0$, is a solid u such that $\Delta(u) \leqslant \varepsilon$.

The following statements are easy consequences of the definition.

(a) For all u and v, $\delta(u,v) = \delta(v,u)$.

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(b) The solid u is a point if and only if $\Delta(u) = 0$.

(c) The set of all points in S is a metric space with respect to δ.

(d) If $u' \leqslant u$ and $v' \leqslant v$, then $\delta(u', v') \geqslant \delta(u, v)$.

(e) For all u, u', v, v',

$$\delta(u,v) - \delta(u',v') \leqslant \delta(u,u') + \delta(v,v') + \Delta(u') + \Delta(v').$$

Hence if these solids are ε-points, then

$$|\delta(u,v) - \delta(u',v')| \leqslant \delta(u,u') + \delta(v,v') + 2\varepsilon.$$

Obviously a metric space (M,d) is a space of solids with $S = M$, $\delta = d$, and $\leqslant$ simple equality. In this trivial example, every solid is a point. For a more interesting example obtained from (M,d), let S be the set of all nonempty compact subsets u of M such that $u^{\circ -} = u$, where $^{\circ}$ denotes interior and $^{-}$ denotes closure; define

$$\delta(u,v) = \min\{d(P,Q) : P \in u, Q \in v\},$$

and let $\leqslant$ be set inclusion. We show that $(S, \leqslant, \delta)$ is a space of solids, called the space *associated with* (M,d).

Axiom (1) is clear. Since v is compact, $\Delta(v) < \infty$. To complete the proof of (2), choose $U \in u$, $V_1 \in v$, $W \in w$, $V_2 \in v$ such that

$$\delta(u,v) = d(U,V_1) \text{ and } \delta(w,v) = d(W,V_2).$$

For $\varepsilon > 0$, let

$$D(V_1) = \{P \in v^{\circ} : d(P,V_1) < \varepsilon\} \text{ and } D(V_2) = \{P \in v^{\circ} : d(P,V_2) < \varepsilon\}.$$

Then $v_1 = D(V_1)^{-}$ and $v_2 = D(V_2)^{-}$ are solids contained in v. Choose $V_1' \in v_1$ and $V_2' \in v_2$ such that $\delta(v_1, v_2) = d(V_1', V_2')$. Then

$$d(V_1,V_2) \leqslant d(V_1,V_1') + d(V_1',V_2') + d(V_2',V_2) \leqslant \varepsilon + \delta(v_1,v_2) + \varepsilon$$

for all $\varepsilon > 0$; so $d(V_1, V_2) \leqslant \delta(v_1, v_2) \leqslant \Delta(v)$. Hence,

$$\delta(u,w) \leqslant d(U,W) \leqslant d(U,V_1) + d(V_1,V_2) + d(V_2,W) \leqslant \delta(u,v) + \delta(v,w) + \Delta(v).$$

For (3), it is obvious that $u \leqslant v$ implies $\delta(u', v) = 0$ for all $u' \leqslant u$. Conversely, suppose that $\delta(u', v) = 0$ for all $u' \leqslant u$ but that $u \nleqslant v$. Choose P in $u^{\circ} \setminus v$ and $\varepsilon > 0$ such that $C = \{X \in M : d(X, P) \leqslant \varepsilon\}$ is disjoint from v. Let $B = \{X \in u^{\circ} : d(X, P) < \varepsilon\}$. Then $u' = B^{-}$ is a solid and $u' \cap v = \varnothing$. Then $\delta(u', v) > 0$, a contradiction; hence, $u \leqslant v$.

For (4), given u, v, and ε, choose U and V such that $\delta(u, v) = d(U, V)$, let $D(U) = \{X \in u^{\circ} : d(X, U) < \varepsilon\}$ and $D(V) = \{X \in v^{\circ} : d(X, V) < \varepsilon\}$. Then $u' = D(U)^{-}$ and $v' = D(V)^{-}$ are ε-points and $\delta(u', v') = \delta(u, v)$.

Recall that a metric space (M, d) is *perfect* if it has no isolated points.

It is easy to see that

(f) a metric space is perfect if and only if its associated space of solids has no points.

The association of a space of solids with a metric space is characteristic of all spaces of solids, in the following sense.

THEOREM. *Every space of solids $(S, \leqslant, \delta)$ is isomorphic to a space of solids $(S', \subseteq, \delta')$, where S' is a family of nonempty subsets of a metric space (M, d) and $\delta'[u, v] = \inf\{d(P, Q) : P \in u$ and $Q \in v\}$.*

Proof. Let $\overline{M}$ be the set of all sequences $\overline{P} = (p_n)$ of solids such that $p_{n+1} \leqslant p_n$ for all n and $\lim \Delta(p_n) = 0$. For $\overline{P}$ and $\overline{Q} = (q_n)$ in $\overline{M}$, define $\bar{d}(\overline{P}, \overline{Q}) = \lim \delta(p_n, q_n)$. Since $(\delta(p_n, q_n))$ is increasing and bounded above by $\delta(p_1, q_1) + \Delta(p_1) + \Delta(q_1)$, $\bar{d}$ is defined on $\overline{M} \times \overline{M}$. It is easy to show that $(\overline{M}, \bar{d})$ is a pseudometric space. Let $M = \overline{M}/\mathscr{R}$, where $\overline{P}\mathscr{R}\overline{Q}$ means $\bar{d}(\overline{P}, \overline{Q}) = 0$, and for $P, Q \in M$ let $d(p, Q) = \bar{d}(\overline{P}, \overline{Q})$, where $\overline{P}$ is in the equivalence class P and $\overline{Q}$ is in the equivalence class Q. Then (M, d) is a metric space, called the space *associated with* $(S, \leqslant, \delta)$. Define the relation *belongs to*, $\beta \subseteq M \times S$, as follows: $P\beta u$ means there is a sequence (p_n) in P with $p_1 = u$. Define a function f from S into the set of all subsets of M by

$$f(u) = \{P \in M : P\beta u\}.$$

Let S' be the image of f. We must prove, for all u, v in S: (i) $f(u) \neq \varnothing$, (ii) $u \leqslant v \Rightarrow f(u) \subseteq f(v)$, (iii) $\delta(u, v) = \delta'[f(u), f(v)]$, (iv) f is injective.

(i) By repeated use of (4) we may build a sequence (p_n) such that $p_1 = u$, $p_{n+1} \leqslant p_n$, and $\Delta(p_n) = 1/(n+1)$. Then the equivalence class of this sequence is in $f(u)$.

(ii) If $u \leqslant v$, then any sequence $u = p_1 \geqslant p_2 \geqslant \cdots$ may be extended to $v \geqslant u \geqslant p_2 \geqslant \cdots$.

(iii) Suppose $P\beta u$ and $Q\beta v$. Choose sequences $\overline{P} = (p_n)$ and $\overline{Q} = (q_n)$ with $p_1 = u$, $q_1 = v$. Then

$$\delta(u, v) = \delta(p_1, q_1) \leqslant \bar{d}(\overline{P}, \overline{Q}) = d(P, Q);$$

so

$$\delta(u, v) \leqslant \inf\{d(P, Q) : P\beta u \text{ and } Q\beta v\} = \delta'[f(u), f(v)].$$

Now by (4) we may build $(p'_n), (q'_n)$ such that

$$p'_1 = u,\ q'_1 = v,\ p'_{n+1} \leqslant p'_n,\ q'_{n+1} \leqslant q'_n,\ \Delta(p'_n) \leqslant 1/n,\ \Delta(q'_n) \leqslant 1/n,$$

and

$$\delta(p'_{n+1}, q'_{n+1}) = \delta(p'_n, q'_n) = \delta(u, v).$$

Then these sequences represent equivalence classes P', Q' such that $\delta(u, v) = d(P', Q')$. Thus

$$\delta(u, v) \geqslant \inf\{d(P, Q) : P\beta u \text{ and } Q\beta v\}.$$

Hence $\delta(u, v) = \delta'[f(u), f(v)]$.

(iv) If $u \neq v$, then we may assume $u \nleqslant v$. By (3) there exists $u' \leqslant u$ such that $\delta(u', v) > 0$. Then by (iii),

$$\inf\{d(P, Q) : P\beta u' \text{ and } Q\beta v\} > 0;$$

hence $f(u') \nleqslant f(v)$. But by (ii), $f(u') \leqslant f(u)$; hence $f(u) \neq f(v)$.

FURTHER REMARKS. Since euclidean, elliptic, and hyperbolic geometries may be defined solely in terms of axioms about metric spaces [1], the definition above leads to axiomatizations of geometries without the primitives "point". It remains to reformulate the resulting axioms in a more natural, less convoluted way. As one example, define three ε-points u, v, w in a space of solids $(S, \leqslant, \delta)$ to be *ε-collinear* if *one* of the following holds:

$$\delta(u, v) + \delta(v, w) - \delta(u, w) < 3\varepsilon,$$

$$\delta(u, w) + \delta(w, v) - \delta(u, v) < 3\varepsilon,$$

$$\delta(v, u) + \delta(u, w) - \delta(v, w) < 3\varepsilon.$$

Then the axiom

(A) *there exist three noncollinear points*

holds in the associated metric space (M, d) if and only if the axiom

(A′) *for some positive ε there exist three ε-points that are not ε-collinear,*

holds in $(S, \leqslant, \delta)$.

Proof. Suppose that P, Q and T are noncollinear elements of M; let (p_n), (q_n) and (t_n) be in P, Q and T. Then there exists $\varepsilon > 0$ such that

$$d(P,T) + d(T,Q) - d(P,Q) > 3\varepsilon,\ d(P,Q) + d(Q,T) - d(P,T) > 3\varepsilon$$

and

$$d(Q,P) + d(P,T) - d(Q,T) > 3\varepsilon.$$

It follows that there exists $m \in N$ such that

$$\Delta(p_n) < \varepsilon,\ \Delta(q_n) < \varepsilon,\ \Delta(t_n) < \varepsilon,\ \delta(p_n, q_n) + \delta(q_n, t_n) - \delta(p_n, t_n) > 3\varepsilon,$$
$$\delta(p_n, t_n) + \delta(t_n, q_n) - \delta(p_n, q_n) > 3\varepsilon \text{ and } \delta(q_n, p_n) + \delta(p_n, t_n) - \delta(q_n, t_n) > 3\varepsilon$$

for every $n \geqslant m$. This proves that p_m, q_m and t_m are not ε-collinear ε-points.

Conversely, suppose that there exist three ε-points u, v and w that are not ε-collinear. Thus, if $P\beta u$, $Q\beta v$ and $T\beta w$, we have that

$$d(P,T) + d(T,Q) - d(P,Q) > 0.$$

Indeed, observe that

$$\delta(u,w) + \delta(w,v) - \delta(u,v) - 2\varepsilon \geqslant \varepsilon.$$

Now, by (iii), there exist $U\beta u$, $U'\beta u$, $V\beta v$, $V'\beta v$, $W\beta w$, $W'\beta w$ such that

$$\delta(u,w) = d(U,W),\ \delta(w,v) = d(W',V),\ \delta(u,v) = d(U',V').$$

Then

$$d(P,T) + d(T,Q) - d(P,Q) \geqslant \delta(u,w) + \delta(w,v) - d(P,Q)$$
$$\geqslant \delta(u,w) + \delta(w,v) - [d(P,U') + d(U',V') + d(V',Q)]$$
$$\geqslant \delta(u,w) + \delta(w,v) - [\varepsilon + \delta(u,v) + \varepsilon] = \delta(u,w) + \delta(w,v) - \delta(u,v) - 2\varepsilon \geqslant \varepsilon > 0.$$

In the same manner one proves that

$$d(P,Q) + d(Q,T) - d(P,T) > 0$$

and that

$$d(Q,P) + d(P,T) - d(Q,T) > 0.$$

In conclusion, P, Q and T are noncollinear.

Axiom A can also be reformulated with no mention of points at all. Define three solids u, v, w to be *collinear* if

$$\delta(u,v) + \delta(v,w) - \delta(u,w) < 3m$$

or

$$\delta(u,w) + \delta(w,v) - \delta(u,v) < 3m$$

or

$$\delta(v,u) + \delta(u,w) - \delta(v,w) < 3m,$$

where $m = \max[\Delta(u), \Delta(v), \Delta(w)]$. Then A holds in (M, d) if and only if

(A″) *there exist three noncollinear solids*
holds in $(S, \leqslant, \delta)$.

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CENTROSYMMETRIC (CROSS-SYMMETRIC) MATRICES, THEIR BASIC PROPERTIES, EIGENVALUES, AND EIGENVECTORS

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1. Introduction. For a number of decades symmetric matrices over the real field have been studied intently by every beginning linear algebra student as a class of matrices which are defined by the property of being symmetric about their main diagonal or by the fact that you can interchange the rows and columns of a given symmetric matrix and it remains unchanged. In [1] (p. 124) A. C. Aitken defines a centrosymmetric matrix P and this definition coincides with the definition given by Graybill of a cross-symmetric matrix [6] (p. 361).

DEFINITION 1. An $n \times n$ matrix P over the real field is centrosymmetric if

$$P_{i,j} = P_{n-i+1,n-j+1}, \quad \text{for } 1 \leqslant i, j \leqslant n.$$

A closer look at Definition 1 reveals that a centrosymmetric matrix is nothing more than a square matrix which is symmetric about its center.

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b & c \\ d & e & d \\ c & b & a \end{pmatrix}$$

are examples of centrosymmetric matrices. This type of matrix is very important in the study of certain types of Markov Processes because they turn out to be transition matrices for the process. As we look at centrosymmetric matrices we will find that they have many interesting properties, comparable in some ways with symmetric matrices, but yet in some ways they are very different from symmetric matrices. For instance; it will be shown that they form an algebra.

2. Historical Notes. M. Iosifescu points out in [7] that "The concept of Markov dependence appears in an explicit form in 1906 in a paper [12] of the Russian mathematician A. A. Markov (1856–1922). In a series of papers starting with [12] he studied various properties of sequences of dependent random variables which in his honor are nowadays called finite Markov chains.

Almost at the same time, studying the problem of shuffling the card deck, the French

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