GRADED INCLUSION AND POINT-FREE GEOMETRY

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Abstract. Inspired to some researches of A. N. Whitehead, we propose an approach to point-free geometry based on the notions of *"region"* and of *"graded inclusion"* between regions.

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1. Introduction

The possibility of considering a geometry in which the notion of point is not assumed as a primitive was extensively examined by A. N. Whitehead in *An Inquiry Concerning the Principles of Natural Knowledge* (1919) and in *The Concept of Nature* (1920) where the primitives are the *regions* and the *inclusion relation* between regions. As a matter of fact, as observed by Casati and Varzi in [2], these books can be a basis for a *"mereology"* i.e. an investigation about the set theoretical *part-whole* relation, rather than about a point-free geometry. So, it is not surprising the fact that, later, in *Process and Reality* (1929) Whitehead proposed a different approach, topological in nature, in which the primitives are the *regions* and the *connection relation*, that is the relation between two regions that either overlap or have at least a common boundary point. The inclusion relation is defined by setting, given two regions x and y, $x \subseteq y$ provided that any region connected with x is necessarily connected with y. Also, Whitehead defines the points, the lines and all the "abstract" geometrical entities whose dimension is different from the dimension of the space. The notion abstraction process is the basic tool to do this, where an abstraction process is a suitable sequence of nested regions.

The aim of this note is to show that by admitting a "graded" inclusion relation the initial inclusion-based approach of Whitehead is possible. In fact, we will consider structures as $(Re,incl)$ where Re is a set whose elements we call *regions* and where $incl:Re\times Re\to[0,1]$ is a fuzzy relation, i.e. a two places function such that the number $incl(x,y)$ represents the degree at which we can consider the region x contained in the region y. We call *graded inclusion spaces* these structures. A suitable definition of the abstraction processes enables us to define the points and the distance between points. So, we associate any graded inclusion space with a metric space (M,δ). In accordance with the results of L. M. Blumenthal, we can impose suitable axioms to $(Re,incl)$ to obtain that (M,δ) is a Euclidean metric space. This gives a point-free axiomatization of the Euclidean geometry based on the graded inclusion as in the primitive Whitehead's program.

2. Preliminaries

The triangular norms where introduced as a basic tool for probabilistic metric space theory and as a basis of several multivalued logics to interpret the logical connective *AND* (see [7] and [3]).

Definition 2.1. A *continuous triangular norm* is a continuous commutative and associative operation $*$ in the complete lattice [0,1] such that

1. $x*1 = x$
2. $*$ is isotone in both arguments.

Together with the operation $*$ we also consider in [0,1] the lattice operations $\wedge$ and $\vee$, i.e. the minimum and the maximum operations. Also, we associate any continuous triangular norm with the *implication* operation defined by setting

$$x \to y = Sup\{z \in [0,1] : x*z \leq y\},$$

and an *equivalence* operation defined by

$$x \leftrightarrow y = (x \to y)*(y \to x).$$

The following are the main examples of a continuous triangular norm:

The *minimum* $\wedge$: in such a case we have that

$x \to y = 1$ if $x \leq y$ and $x \to y = y$ otherwise
$x \leftrightarrow y = 1$ if $x = y$ and $x \leftrightarrow y = x \wedge y$ otherwise.

The *product*: in such a case we have that

$x \to y = 1$ if $x \leq y$ and $x \to y = y/x$ otherwise
$x \leftrightarrow y = 1$ if $x = y$ and $x \leftrightarrow y = (x \wedge y)/(x \vee y)$ otherwise.

The *Lukasiewicz norm* defined by setting $x*y = x+y-1$ if $x+y-1 \geq 0$ and $x*y = 0$ otherwise: in such a case

$x \to y = 1$ if $x \leq y$ and $x \to y = y-x+1$ otherwise
$x \leftrightarrow y = 1-|x-y|$.

Observe that the continuity of $*$ does not imply the continuity of the associated operations $\to$ and $\leftrightarrow$. The following proposition lists some basic properties of a continuous triangular norm (see [3] and [7]).

Proposition 2.2. *Let $*$ be a continuous triangular norm, $(x_i)_{i \in I}$ be a family of elements in* [0,1] *and x, y, z elements in* [0,1]. *Then*

(*i*) $x \to x = 1$,
(*ii*) $(x \to y)*(y \to z) \leq x \to z$,
(*iii*) $x \to y = 1$ and $y \to x = 1 \Rightarrow x = y$
(*iv*) $x \to y = 1 \Leftrightarrow x \leq y$
(*v*) $x*z \leq y \Leftrightarrow z \leq x \to y$
(*vi*) $(z \to y)*z \leq y$
(*vii*) $Sup_{i \in I}(x*x_i) = x*(Sup_{i \in I} x_i)$,
(*viii*) $Sup_{i \in I}(x \to x_i) \leq x \to (Sup_{i \in I} x_i)$,
(*ix*) $Sup_{i \in I}(x_i \to x) \leq (Inf_{i \in I} x_i) \to x$,
(*x*) $Inf_{i \in I}(x*x_i) \geq x*(Inf_{i \in I} x_i)$,
(*xi*) $Inf_{i \in I}(x \to x_i) = x \to (Inf_{i \in I} x_i)$,
(*xii*) $Inf_{i \in I}(x_i \to x) = (Sup_{i \in I} x_i) \to x$.

Moreover,

(*xiii*) $x \leftrightarrow x = 1$,
(*xiv*) $x \leftrightarrow y = 1 \Leftrightarrow x = y$
(*xv*) $(x \leftrightarrow y)*(y \leftrightarrow z) \leq x \leftrightarrow z$
(*xvi*) $x \leftrightarrow y = (x \to y) \wedge (y \to x)$.

A basic class of triangular norms is the following. As usual, for any $x \in [0,1]$, x^n is defined by the equations $x^1 = x$ and $x^{n+1} = x*x^n$.

Definition 2.3. A continuous triangular norm $*$ is *Archimedean* if, for any $x \neq 1$, $lim_{n \to \infty} x^n = 0$.

The usual product and the Lukasiewicz norm are examples of Archimedean continuous triangular norms. The minimum is an example of continuous norm which is not Archimedean. There is a very interesting characterization of the Archimedean triangular norms. In the following, we consider the extended interval $[0,\infty]$ and we assume that $x+\infty = \infty+x = \infty$ and that $x \leq \infty$ for any $x \in [0,\infty]$.

Definition 2.4. A map $f : [0,1] \to [0,\infty]$ is an *additive generator* provided that f is a continuous strictly decreasing function such that $f(1) = 0$.

The *pseudoinverse* $f^{[-1]} : [0,\infty] \to [0, 1]$ of an additive generator f is defined by setting, for any $y \in [0,\infty]$,

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in f([0, 1]), \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate to see that $f^{[-1]}$ is order-reversing and that $f^{[-1]}(0) = 1$ and $f^{[-1]}(\infty) = 0$. Moreover, we have that, for any $x \in S$,
- $f^{[-1]}(f(x)) = x$
- $f(f^{[-1]}(x)) = x \wedge f(0)$.

Theorem 2.5. *An operation* $*$: $[0,1]\times[0,1]\to[0,1]$ *is a continuous Archimedean triangular norm if and only if an additive generator* f : $[0,1] \to [0, \infty]$ *exists such that*

$$x * y = f^{[-1]}(f(x) + f(y)) \tag{2.1}$$

for all x, y in $[0, 1]$. *In such a case*

$$x \to y = f^{[-1]}(f(y) - f(x)) \tag{2.2}$$

and

$$x \leftrightarrow y = f^{[-1]}(|f(y) - f(x)|) \tag{2.3}$$

Proof. See [3]. □

As an example, assume that $f(x) = -log(x)$ (where, as usual, we set $-log(0) = \infty$). Then, $f^{[-1]}(y) = e^{-y}$ (where, as usual, we set $e^{-\infty} = 0$) and therefore

$$x * y = e^{-(-log(x) - log(y))} = e^{log(x \cdot y)} = x \cdot y,$$

i.e. the triangular norm $*$ generated by f is the usual product in [0,1]. Assume that $f(x) = 1-x$. Then, since $f([0,1]) = [0,1]$, we have that $f^{[-1]}(x) = f(x)$ if $x \in [0,1]$ and $f^{[-1]}(x) = 0$ otherwise. Consequently, in the case $x+y-1 \in [0,1]$ we have that $x*y = 1-(1-x+1-y) = 1-1+x-1+y = x+y-1$, while, if $x+y-1 \notin [0,1]$, $x*y = 0$. So, in such a case $*$ coincides with the Lukasiewicz triangular norm.

Observe that any triangular norm $*$ defines a first order multivalued logic in which the conjunction is interpreted by $*$ and the implication and the equivalence are interpreted by the operations $\to$ and $\leftrightarrow$, respectively. In such a logic 0 is interpreted as the truth-value *false* and 1 as the truth value *true*. Moreover, the negation is usually interpreted by the map $1-x$, and the disjunction by the *conorm* $\oplus$ defined by setting

$$x \oplus y = 1-(1-x)*(1-y).$$

Finally, the existential quantifier is interpreted by the least upper bound and the universal quantifier by the greatest lower bound. In this paper we don't refer to multivalued logic in a formal way but we consider it as an euristic tool to translate a classical notion into a corresponding "graded" notion.

3. Graded preorders and orders.

Let $*$ be a triangular norm and $ord : S \times S \to [0,1]$ a map. Then we are interested to the following properties:

A1	$ord(x,x) = 1$,	(reflexivity)
A2	$ord(x,y)*ord(y,z) \leq ord(x,z)$,	($*$-transitivity)
A3	$ord(x,y) = ord(y,x) = 1 \Rightarrow x = y$,	(antisimmetry)
S	$ord(x,y) = ord(y,x)$.	(simmetry)

where, $x, y, z \in S$.

These properties are suggested by corresponding ones for binary relations in classical set theory. For example, a relation $\leq$ is reflexive if it satisfies the axiom $\forall x(x \leq x)$. Interpret this sentence in a multivalued logic in which $\leq$ denotes a graded relation $ord : S \times S \to [0,1]$. Then, since the universal quantifier is interpreted by the least upper bound, $\forall x(x\leq x)$ is valued 1 if and only if $Inf\{ord(x,x) : x\in S\} = 1$ and therefore if and only if *A1* holds for any $x \in S$. Also, the relation $\leq$ is transitive if the axiom $\forall x\forall y\forall z((x\leq y)\wedge(y\leq z)\rightarrow(x\leq z))$ is satisfied. Taking in account the fact that $\lambda\rightarrow\mu = 1$ if and only if $\lambda\leq\mu$, this holds if and only if *A2* holds.

Definition 3.1. A map $ord : S\times S\rightarrow[0,1]$ is called:
- a $*$-*preorder* if it satisfies *A1* and *A2*,
- a $*$-*order*, provided that it satisfies *A1*, *A2* and *A3*,
- a $*$-*similarity*, provided that it satisfies *A1*, *A2* and *S*.
- a *strict* $*$-*similarity*, provided that it satisfies *A1*, *A2*, *A3* and *S*.

We say that *ord* is *crisp* provided that it assumes values only in the Boolean set $\{0,1\}$. Trivially, the crisp $*$-preorders ($*$-orders, $*$-similarities, strict $*$-similarities) coincide with the characteristic functions of the preorders (orders, equivalence relations, identity, respectively). Then, the proposed notions extend the classical ones.

Proposition 3.2. *Given a* $*$-*preorder* ($*$-*order*) (S, ord), *the relation* $\leq$ *defined by setting*

$$x\leq y \Leftrightarrow ord(x,y) = 1 \tag{3.1}$$

is a preorder (*order*) *we call the preorder* (*order*) *relation associated with ord. Conversely, let* $\leq$ *be a pre-order* (*order*) *relation, then its characteristic function is a* $*$-*preorder* ($*$-*order*).

If *ord* is a (strict) $*$-similarity, then the induced order is an equivalence relation (the identity relation).

Proposition 3.3. *Any* $*$-*preorder* $ord : S\times S\rightarrow[0,1]$ *is order-reversing with respect to the first variable and order-preserving with respect to the second variable.*

Proof. Assume that $x'\leq x$, then $ord(x',y) \geq ord(x',x)*ord(x,y) = ord(x,y)$. Assume that $y'\leq y$, then $ord(x,y)\geq ord(x,y')*ord(y',y) = ord(x,y')$. □

It is well known that any preorder $\leq$ in a set S induces an equivalence relation $\equiv$ defined by setting $x\equiv y$ provided that $x\leq y$ and $y\leq x$. Also, in the quotient $S/\equiv$ we define an order relation by setting $[x]\leq[y]$ if $x\leq y$. We can extend this to the $*$-orders by the following proposition where $\wedge$ denotes the minimum operation.

Proposition 3.4. *Let* $ord : S\times S\rightarrow[0,1]$ *be a* $*$-*preorder and define the graded relation* $eq : S\times S\rightarrow[0,1]$ *by setting*

$$eq(x,y) = ord(x,y)\wedge ord(y,x) \tag{3.2}$$

for any $x,y \in S$. *Then eq is a* $*$-*similarity relation. Let* $\equiv$ *be the equivalence relation associated with eq and set, in the quotient* $S/\equiv$, $ord([x],[y]) = ord(x,y)$. *Then* $(S/\equiv, ord)$ *is a* $*$-*order we call the quotient of* (S,ord).

Proof. It is immediate that $eq(x,x) = 1$ and $eq(x,y) = eq(y,x)$. Also, from the trivial inequalities

$$ord(y,z) \geq ord(z,y)\wedge ord(y,z) \ ; \ ord(z,x) \geq ord(x,z)\wedge ord(z,x)$$

and the fact that $*$ is order-preserving, we have that

$$ord(y,z)*ord(z,x) \geq [ord(z,y)\wedge ord(y,z)]*[ord(x,z)\wedge ord(z,x)].$$

In a similar way we prove that

$$ord(x,z)*ord(z,y) \geq [ord(x,z)\wedge ord(z,x)]*[ord(z,y)\wedge ord(y,z)].$$

Then, since *ord* is transitive,

$$eq(x,y) = ord(x,y)\wedge ord(y,x) \geq [ord(x,z)*ord(z,y)]\wedge[ord(y,z)*ord(z,x)]$$
$$\geq [ord(x,z)\wedge ord(z,x)]*[ord(z,y)\wedge ord(y,z)] = eq(x,z)*eq(z,y).$$

The remaining part of the proposition is obvious. □

4. Abstraction processes

The aim of this note is to represent the notion of region and graded inclusion between two regions in a geometrical space. In accordance, in the following we prefer the notation (*Re*,*incl*) to denote a $*$–preorder structure, we call *regions* the elements in *Re* and *graded inclusion* the function *incl*. Whitehead defined the points by the *"abstraction processs"* i.e. by suitable order-reversing sequences of regions. Then, in order to simulate Whitehead's approach, we have to define an analogous of the abstractions processes.

Definition 4.1. The *degree of pointlikeness* of an element $x \in Re$ is the number

$$pl(x) = Inf\{eq(x',x) : x'\leq x\}. \tag{4.1}$$

We can interpret $pl(x)$ as the degree of validity of the claim *"any subregion of x coincides with x"* by which we can define the notion of minimal element in an ordered set. This in accordance with the fact that, if *ord* is a $*$-order, then $pl(x) = 1$ if and only if x is a minimal element with respect to the order relation induced by *incl*. If *ord* is a $*$-similarity, then $pl(x) = 1$ for any region x. The following aree equivalent definitions of the degree of pointlikeness.

Proposition 4.2. *For any $x\in Re$, we have that*

$$pl(x) = Inf\{incl(x,x') : x'\leq x\}. \tag{4.2}$$

$$pl(x) = Inf\{incl(x_1,x_2) : x_1\leq x,\ x_2\leq x\}. \tag{4.3}$$

$$pl(x) = Inf\{eq(x_1 ,x_2) : x_1\leq x,\ x_2\leq x\}. \tag{4.4}$$

Proof. Equation (4.2) is immediate. To prove (4.3) observe that

$$pl(x) = Inf\{incl(x,x_2) : x_2\leq x\} \geq Inf\{incl(x_1,x_2) : x_1\leq x,\ x_2\leq x\}$$

and that, since by the $*$-transitivity,

$$incl(x_1,x_2) \geq incl(x_1,x)*incl(x,x_2) = 1*incl(x,x_2) = incl(x,x_2),$$

it is

$$Inf\{incl(x_1,x_2) : x_1\leq x,\ x_2\leq x\} \geq Inf\{incl(x,x_2) : x_2\leq x\} = pl(x).$$

It is evident that (4.4) is equivalent to (4.3). □

Definition 4.3. A sequence of regions $<p_n>_{n\in N}$ of (*Re*,*incl*) is called an *abstraction process* if

a) $lim_{n\to\infty}pl(p_n) = 1$;

b) $\forall\varepsilon<1\ \ \exists m : h\geq k\geq m \Rightarrow ord(p_h,p_k) >\varepsilon$.

We denote by *Pr* the class of abstraction processes.

The following axiom says that if the regions x and y are (approximately) points, then the graded inclusion is (approximately) symmetric.

***A4*)** $$pl(x)*pl(y) \leq (incl(x,y)\leftrightarrow incl(y,x)).$$

If *ord* is a $*$-similarity, then such an axiom is satisfied in a trivial way. Observe that we can rewrite *A4*) as follows:

$$pl(x)*pl(y)*incl(x,y)\leq incl(y,x).$$

Proposition 4.4. *Assume A4, and let $<p_n>_{n\in N}$ be a sequence of regions such that $lim_{n\to\infty}pl(p_n) = 1$. Then the following are equivalent:*

i) *$<p_n>_{n\in N}$ is an abstraction process*

ii) $\forall\varepsilon<1\ \ \exists m : h\geq m,\ k\geq m \Rightarrow incl(p_h,p_k) >\varepsilon$.

iii) $\forall\varepsilon<1\ \ \exists m : h\geq m,\ k\geq m \Rightarrow eq(p_h,p_k) >\varepsilon$.

Proof. Let $<p_n>_{n\in N}$ be an abstraction process, and let $\varepsilon<1$. Also, let δ be such that $\delta^3\geq \varepsilon$. Then a natural number m exists such that $incl(p_h,p_k)\geq \delta$, $pl(p_h)\geq\delta$ and $pl(p_k)\geq\delta$ for any $h\geq k\geq m$. Since

$$incl(p_k,p_h)\geq pl(p_k)*pl(p_h)*incl(p_h,p_k)\geq \delta^3\geq\varepsilon,$$

ii) follows. The remaining part of the proposition is trivial. □

5. Graded inclusion spaces and associated metric spaces.

We are now able to define the graded inclusion spaces. In the next sections we assume that $*$ is an Archimedean triangular norm whose generator is f.

Definition 5.1. We call *graded inclusion space* any model $(Re,incl)$ of *A1*, *A2*, *A3*, *A4* such that

A5) $Pr\neq\emptyset$,

i.e. such that an abstraction process exists.

Trivial examples of graded inclusion spaces are given by the strict $*$-similarities. In fact, in such a case any constant sequence $<p_n>_{n\in N}$ of regions is an abstraction process. These spaces are not interesting from our point of view since all the regions are points, in a sense. In any graded inclusion space we can define a $*$-similarity in the set *Pr* of abstraction processes.

Proposition 5.2. *Define the map sim* : $Pr\times Pr\to[0,1]$ *by setting*

$$sim(<p_n>_{n\in N},<q_n>_{n\in N}) = lim_{n\to\infty}\, incl(p_n,q_n). \qquad (5.1)$$

for any $<p_n>_{n\in N}$ *and* $<q_n>_{n\in N}$ *in Pr. Then the structure* (Pr,sim) *is a* $*$*-similarity.*

Proof. To prove that for any pair $<p_n>_{n\in N}$ and $<q_n>_{n\in N}$ of elements in *Pr* the sequence $<incl(p_n,q_n)>_{n\in N}$ is convergent, we at first prove that

$$\forall\delta \text{ s.t. } 1>\delta>0,, \exists m(h\geq m \text{ and } k\geq m \Rightarrow incl(p_h,q_h)\leftrightarrow incl(p_k,q_k)\geq\delta). \qquad (5.2)$$

Indeed, since

$$incl(p_h,q_h)\geq incl(p_h,p_k)*incl(p_k,q_k)*incl(q_k,q_h)$$

by *v*) of Proposition 2.2 and by *A4*, we have that

$$incl(p_k,q_k)\to incl(p_h,q_h)\geq incl(p_h,p_k)*incl(q_k,q_h)\geq incl(p_k,p_h)*incl(q_k,q_h)*pl(p_k)*pl(p_h)$$
$$\geq incl(p_k,p_h)*incl(q_k,q_h)*pl(p_k)*pl(p_h)*pl(q_k)*pl(q_h)$$

Likewise, we prove that

$$incl(p_h,q_h)\to incl(p_k,q_k)\geq incl(p_k,p_h)*incl(q_k,q_h)*pl(q_k)*pl(q_h)*pl(p_k)*pl(p_h).$$

Consequently,

$$incl(p_h,q_h)\leftrightarrow incl(p_k,q_k)\geq incl(p_k,p_h)*incl(q_k,q_h)*pl(p_k)*pl(p_h)*pl(q_k)*pl(q_h).$$

Given $1>\delta>0$, since $lim_{\gamma\to1}\gamma^6=1$, $\gamma<1$ exists such that $\delta\leq\gamma^6$. Let m be such that for any $i\geq m$ and $j\geq m$,

$$incl(p_i,p_j)\geq\gamma,\ pl(p_j)\geq\gamma,\ incl(q_i,q_j)\geq\gamma,\ pl(q_j)\geq\gamma.$$

Then $incl(p_h,q_h)\leftrightarrow incl(p_k,q_k)\geq\gamma^6\geq\delta$ for any $h\geq m$, $k\geq m$ and this proves (5.2).

Now, by (2.3), (5.2) is equivalent with

$$\forall\delta>0,\ 1>\delta,\ \exists m\ h\geq m \text{ and } k\geq m\Rightarrow f^{[-1]}(|f(incl(p_h,q_h))-f(incl(p_k,q_k))|)\geq\delta.$$

and, since $|f(incl(p_h,q_h))-f(incl(p_k,q_k))|$ is an element of the interval $f([0,1])$, this entails that

$$\forall\delta>0,\ 1>\delta\ \exists m\ h\geq m \text{ and } k\geq m\Rightarrow |f(incl(p_h,q_h))-f(incl(p_k,q_k))|\leq f(\delta).$$

Then, since for any $\varepsilon>0$ there is $\delta<1$ such that $f(\delta)\leq\varepsilon$, we have that

$$\forall\varepsilon>0\ \exists m\ h\geq m \text{ and } k\geq m\Rightarrow |f(incl(p_h,q_h))-f(incl(p_k,q_k))|\leq\varepsilon.$$

This proves that $<f(incl(p_n,q_n))>_{n\in N}$ is convergent. Since f^{1} is continuous in the interval $f([0,1])$, we can conclude that $<incl(p_n,q_n)>_{n\in N}$ is convergent, too.

It is immediate that *sim* is reflexive. To prove the $*$-transitivity, observe that

$$sim(<p_n>,<r_n>) = lim_{n\to\infty}\, incl(p_n,q_n)\geq lim_{n\to\infty}\, incl(p_n,q_n)*(incl(q_n,r_n))$$
$$= lim_{n\to\infty}\, incl(p_n,q_n)*lim_{n\to\infty}\,(incl(q_n,r_n)) = sim(<p_n>,<q_n>)*sim(<q_n>,<r_n>).$$

To prove that *sim* is symmetric, observe that

$$sim(<p_n>,<q_n>) = lim_{n\to\infty} incl(p_n,q_n) \geq lim_{n\to\infty} incl(q_n,p_n)*pl(p_n)*pl(q_n)).$$

Then, since $lim_{n\to\infty}pl(p_n) = lim_{n\to\infty}pl(q_n)) = 1$,

$$sim(<p_n>,<q_n>) \geq lim_{n\to\infty} d(q_n,p_n) = sim(<q_n>,<p_n>).$$ □

In the case of the strict $*$-similarities the space (Pr,sim) is an extension of (Re,ord). As an immediate consequence of Proposition 5.2, we have the following proposition.

Proposition 5.3. *Let (Re,ord) be a $*$-graded inclusion space and let $f : [0,1]\to[0,\infty]$ be an additive generator of $*$. Then the map $d : Pr\times Pr\to R^+$ obtained by setting*

$$d(<p_n>_{n\in N}, <q_n>_{n\in N}) = f(sim(<p_n>_{n\in N}, <q_n>_{n\in N})), \tag{5.3}$$

defines a pseudo-metric space (Pr,d).

Proof. Trivially, d is symmetric. Moreover,

$$\begin{aligned} d(<p_n>_{n\in N}, <q_n>_{n\in N}) &= f(sim(<p_n>_{n\in N}, <q_n>_{n\in N})) \\ &\leq f(sim(<p_n>_{n\in N}, <r_n>_{n\in N})*sim(<r_n>_{n\in N}, <q_n>_{n\in N})) \\ &= f(f^{[-1]}(f(sim(<p_n>_{n\in N}, <r_n>_{n\in N}))+f(sim(<r_n>_{n\in N}, <q_n>_{n\in N}))) \\ &\leq f(sim(<p_n>_{n\in N}, <r_n>_{n\in N}))+f(sim(<r_n>_{n\in N}, <q_n>_{n\in N})) \\ &= d(<p_n>_{n\in N}, <r_n>_{n\in N})+ d(<r_n>_{n\in N}, <q_n>_{n\in N}). \end{aligned}$$ □

Definition 5.4. We call *metric space associated with a graded inclusion space* (Re,$incl$) the quotient (M,δ) of the pseudo-metric space (Pr,d). We call *point* any element in M.

Then, the metric space (M,δ) associated with an inclusion space (Re,$incl$) is obtained

- by starting from the class Pr of point-representing sequences ;
- by setting M equal to the quotient of Pr modulo the equivalence relation $\equiv$ defined by
$$<p_n>_{n\in N} \equiv <q_n>_{n\in N} \Leftrightarrow lim_{n\to\infty} incl(p_n,q_n) = 1 ;$$
- by defining $\delta : M \times M \to R^+$ by the equation,
$$\delta(P,Q) = lim_{n\to\infty} f(incl(p_n,q_n)). \tag{5.4}$$
where $P = [<p_n>_{n\in N}]$ and $Q = [<q_n>_{n\in N}]$ are elements in M.

Notice that if $D(M)$ is the diameter of this space, then, in accordance with (5.4), $D(M) \leq f(0)$.

We conclude this section by noticing that we can also consider a different notion of abstraction process which is closer to Whitehead's definition. Indeed, given a graded inclusion space (Re,$incl$), we call *nested-representing sequence* any order-reversing sequence $<p_n>_{n\in N}$ of regions such that

$$lim_{n\to} pl(p_n) = 1.$$

We denote by Nr the class of the nested-representing sequences. In the case of a strict $*$-similarity, Nr coincides with Re. Obviously, any nested-representing sequence is an abstraction process. These sequences define a metric space (M',δ') where $M' = \{[<p_n>_{n\in N}] \in M : <p_n>_{n\in N} \in Nr\}$ and δ' is the restriction of δ to M'. We call *small metric space associated with* (Re,$incl$) such a metric space.

6. Examples of graded inclusion structures.

Any continuous triangular norm $*$ defines a very simple $*$-graded inclusion structure in [0,1]. Indeed, let $\to$ be the implication associated with $*$. Then from Proposition 2.2 we have that ([0,1],$\to$) is a $*$-order whose induced order is the usual one in [0,1]. Moreover,

$$pl(x) = Inf\{x\to x' : x'\leq x\} = x\to Inf_{x'\leq x}x' = x\to 0$$

i.e. pl coincides with the *strong negation*. Also, since $x\to 0 \leq x\to y$, we have that

$$pl(x)*pl(y)*(y\to x) \leq x\to y$$

and therefore *A4* is satisfied. Finally, the sequence constantly equal to 0 is an abstraction process and A5 is satisfied, too. Thus, ([0,1],$\to$) is a $*$-graded inclusion structure. Now, observe that in the case of the minimum and the usual product, we have that $pl(x) = 1$ if $x = 0$ and $pl(x) = 0$ otherwise. This means that the only abstraction process is the sequence constantly equal to 0. In the case of the Lukasiewicz norm we have that $pl(x) = 1\text{-}x$ and therefore any sequence converging to 0 is an abstraction process. In all the cases in the associated metric space there is only a point.

Let $(S,\leq)$ be an ordered set with at least a minimal element and let *incl* be the characteristic function of $\leq$, i.e., $incl(x,y) = 1$ if $x\leq y$ and $incl(x,y) = 0$ otherwise. Then $(S,incl)$ is a $*$-graded inclusion with respect to any triangular norm $*$ whose induced order coincides with $\leq$. Also, $pl(x) = 1$ if x is a minimal element and $pl(x) = 0$ otherwise. It is immediate to see that both the metric spaces (M,δ) and (M',δ') associated with $(S,incl)$ coincide with the discrete metric in the set of minimal elements of $(S,\leq)$. In particular, the class of nonempty subsets of a given set S ordered with respect to the inclusion relation defines a $*$-graded inclusion whose associated metric space is the discrete metric in S.

The notion of *excess* enables us to obtain examples of graded inclusion structures that are more interesting from a geometrical point of view. As usual, given a metric space (M,δ), $P\in M$ and x, y nonempty subsets of M, we set

$$\delta(P,x) = Inf\{\delta(P,Q) : Q \in x\}. \tag{6.1}$$

$$m(x,y) = Inf\{\delta(P,Q) : P\in x,\ Q\in y\} \tag{6.2}$$

$$D(x) = Sup\{\delta(P,P') : P\in x,\ P'\in x\}. \tag{6.3}$$

It is immediate to prove that, for any P, $Q \in M$ and x nonempty subset,

$$\delta(P,x) \leq \delta(P,Q) + \delta(Q,x). \tag{6.4}$$

Also, we define the *excess function* e_δ by setting,

$$e_\delta(x,y) = Sup\{\delta(P,y) : P\in x\}. \tag{6.5}$$

Recall that by setting $\delta_H(x,y) = e_\delta(x,y)\vee e_\delta(y,x)$ we obtain the famous Hausdorff distance δ_H.

Proposition 6.1. *Let x, y be nonempty subsets of M. Then*

$$|e_\delta(x,y) - e_\delta(y,x)| \leq Max\{D(x), D(y)\}. \tag{6.6}$$

$$e_\delta(x,z) \leq e_\delta(x,y) + e_\delta(y,z). \tag{6.7}$$

Proof. To prove (6.6) observe that, for any P, $P' \in x$,

$$\delta(P,y) \leq \delta(P,P')+\delta(P',y) \leq D(x) + \delta(P',y)$$

and therefore

$$e_\delta(x,y) = Sup\{\delta(P,y) : P\in x\} \leq D(x)+ Inf\{\delta(P',y) : P'\in x\} = D(x) + m(x,y) \leq D(x) + e_\delta(y,x).$$

Then $e_\delta(x,y)\text{-}e_\delta(y,x) \leq D(x) \leq Max\{D(x), D(y)\}$. In the same way we prove that $e_\delta(y,x)\text{-}e_\delta(x,y) \leq D(y) \leq Max\{D(x),D(y)\}$. Thus, (6.6) follows. To prove (6.7), observe that, given $P\in x$ and $Q\in z$,

$$\delta(P,y) \leq \delta(P,Q) +\delta(Q,y) \leq \delta(P,Q) +e_\delta(z,y)$$

and therefore

$$\delta(P,y) \leq Inf_{Q\in z}\, \delta(P,Q) + e_\delta(z,y) = \delta(P,z) + e_\delta(z,y).$$

Consequently,

$$e_\delta(x,y) = Sup\{\delta(P,y) : P\in x\}\leq Sup\{\delta(P,z) + e_\delta(z,y) : P\in x\}$$
$$= Sup\{\delta(P,z) : P\in x\} + e_\delta(z,y) = e_\delta(x,z)+ e_\delta(z,y).$$

□

Denote by $cl : \mathcal{P}(M)\to \mathcal{P}(M)$ and by $int : \mathcal{P}(M)\to \mathcal{P}(M)$ the closure operator and the interior operator, respectively, and define $reg : \mathcal{P}(M)\to \mathcal{P}(M)$ by setting $reg(x) = cl(int(x))$. Then we call *regularly closed set*, in brief *regular set*, any fixed point of *reg*. It is easy to prove that in the class of the closed subsets, the operator *reg* satisfies the following properties

i) $reg(\emptyset) = \emptyset$; ii) $x \subseteq y \Rightarrow reg(x) \subseteq reg(y)$;

iii) $reg(x) \subseteq x$; iv) $reg(reg(x)) = reg(x)$.

The class of regular subsets is an interesting complete Boolean algebra. Basic examples of regular sets are obtained by setting, for any $P\in M$ and $n\in N$,

$$B_n(P) = cl(\{P'\in M : \delta(P',P)<1/n\}). \tag{6.8}$$

Since any point in a regular set x is limit of a sequence of point in $int(x)$, it is $D(x) = D(int(x))$ and, given a point P, $\delta(P,x) = \delta(P,int(x))$.

Theorem 6.2. *Let Re be the class of all nonempty bounded closed regular subsets of* (M,δ) *and let* $f : [0,1]\to[0,\infty]$ *be an additive generator of a triangular norm* $*$. *Also, define incl* : $Re\times Re$ *by setting*

$$incl(x,y) = f^{[-1]}(e_\delta(x,y)). \tag{6.9}$$

Then $(Re,incl)$ *is a* $*$*-graded inclusion space whose associated order is the inclusion relation. If* (M,δ) *is complete and* $D(M)\leq f(0)$, *then the metric space associate with* $(Re,incl)$ is *isometric with* (M,δ).

Proof. *A1* is trivial. To prove *A2*, at first we observe that, for x, y and z positive real numbers,

$$(x+y)\wedge z \leq x\wedge z+y\wedge z. \tag{6.10}$$

Indeed, if $z \leq x$, then since it is also $z \leq x+y$ we have that $(x+y)\wedge z = z \leq z+(y\wedge z) = x\wedge z+y\wedge z$. The same holds in the case $z \leq y$. If $z > x$ and $z > y$, then $(x+y)\wedge z \leq x+y = x\wedge z+y\wedge z$.
Now, by (6.7) and (6.10)

$$e_\delta(x,z)\wedge f(0) \leq (e_\delta(x,y) + e_\delta(y,z))\wedge f(0) \leq e_\delta(x,y)\wedge f(0) + e_\delta(y,z)\wedge f(0)$$

and therefore

$$f(f^{[-1]}(e_\delta(x,z))) \leq f(f^{[-1]}(e_\delta(x,y))) + f(f^{[-1]}(e_\delta(y,z))),$$

i.e.,

$$f(incl(x,z)) \leq f(incl(x,y)) + f(incl(y,z)).$$

Since $f^{[-1]}$ is order-reversing

$$incl(x,z) = f^{[-1]}(f(incl(x,z))) \geq f^{[-1]}(f(incl(x,y)) + f(incl(y,z))) = incl(x,y)*incl(y,z).$$

To prove A3, assume that $incl(x,y) = 1$ and therefore that $f^{[-1]}(e_\delta(x,y)) = 1$, then $f(f^{[-1]}(e_\delta(x,y))) = f(0)\wedge e_\delta(x,y) = 0$ and therefore $e_\delta(x,y) = 0$. Since y is a closed sets, this entails that $x\subseteq y$. This proves both that *A3* is satisfied and that the associated order is the inclusion relation.

To prove *A4*, observe that, for x, y and z positive real numbers,

$$|x-y|\wedge z \geq |x\wedge z - y\wedge z|. \tag{6.11}$$

In fact, assume that and $x \geq y$. Then in the case $z \leq x-y$,

$$|x-y|\wedge z = (x-y)\wedge z = z \geq x\wedge z \geq x\wedge z - y\wedge z = |x\wedge z - y\wedge z|.$$

In the case $z > x-y$ we have that $z \geq x$, and therefore $z\geq y$. So,

$$|x-y|\wedge z = (x-y)\wedge z = x - y = x\wedge z - y\wedge z = |x\wedge z - y\wedge z|.$$

Also, we have

$$f(pl(x)) \geq D(x)\wedge f(0). \tag{6.12}$$

Indeed, for any P and P' in $int(x)$, and for any n such that $B_n(P') \leq x$,

$$\delta(P,B_n(P')) = Inf\{\delta(P,X) : \delta(X,P')<1/n\} \geq Inf\{\delta(P,P')-\delta(X,P') : \delta(X,P')<1/n\} \geq \delta(P,P')-1/n$$

and therefore

$$\begin{aligned} f(pl(x)) &= f(Inf\{incl(x,x') : x'\leq x\}) \\ &= Sup\{f(incl(x,x')) : x'\leq x\} \\ &= Sup\{e_\delta(x,x')\wedge f(0) : x'\leq x\} \\ &= Sup\{e_\delta(x,x') : x'\leq x\}\wedge f(0) \\ &\geq \delta(P,B_n(P'))\wedge f(0) \\ &\geq (\delta(P,P')-1/n)\wedge f(0). \end{aligned}$$

This entail,

$$f(pl(x)) \geq lim_{n\to\infty} (\delta(P,P')-1/n)\wedge f(0) = \delta(P,P')\wedge f(0).$$

Moreover, in accordance with (6.6), (6.10) and (6.12),

$$\begin{aligned} |f(incl(x,y)) - f(incl(y,x))| &= |e_\delta(x,y)\wedge f(0) - e_\delta(y,x)\wedge f(0)| \leq |e_\delta(x,y) - e_\delta(y,x)|\wedge f(0) \\ &\leq (D(x)\vee D(y))\wedge f(0) = D(x)\wedge f(0)\vee D(y)\wedge f(0) \\ &\leq f(pl(x))\vee f(pl(y)) \leq f(pl(x)) + f(pl(y)), \end{aligned}$$

and, since $f^{[-1]}$ is order-reversing,

$$f^{[-1]}(|f(incl(x,y)-f(incl(y,x))|) \geq f^{[-1]}(f(pl(x)) + f(pl(y))).$$

This inequality, by Theorem 2.5, coincides with *A4*.

To prove A5, given any point P in M, observe that,

$$lim_{n\to\infty}pl(B_n(P)) = lim_{n\to\infty}f^{[-1]}(D(B_n(P))) = f^{[-1]}(lim_{n\to\infty}D(p_n)) = f^{[-1]}(0) = 1,$$

so, since $(B_n(P))_{n\in N}$ is order reversing, $(B_n(P))_{n\in N}$ is an abstraction process. This complete the proof that $(Re,incl)$ is a $*$-graded inclusion space.

Assume that $D(M)\leq f(0)$ and let (M',δ') be the metric space associated with $(Re,incl)$. We denote by $i : M \to M'$ the map defined by the equation

$$i(P) = [<B_n(P)>].$$

Then, since it is easy to prove that $\delta(P,P') + 1/n \geq e_\delta(B_n(P),B_n(P')) \geq \delta(P,P') - 1/n$ and therefore that $lim_{n\to\infty}\ e_\delta(B_n(P),B_n(Q)) = \delta(P,Q)$, we have that for any P and Q in M,

$$\begin{aligned}\delta'(i(P),i(Q)) &= lim_{n\to\infty}\ f(incl(B_n(P), B_n(Q)))\\ &= lim_{n\to\infty}\ f(f^{[-1]}(e_\delta(B_n(P),B_n(Q))))\\ &= lim_{n\to\infty}\ e_\delta(B_n(P),B_n(Q))\\ &= \delta(P,Q).\end{aligned}$$

This proves that i is an isometry. To prove that i is surjective, assume that (M,δ) is complete and let P be any point in M', i.e. $P = [<p_n>]$ where $<p_n>$ is an abstraction process. Let, for any $n\in N$, P_n be a point in the set p_n, we claim that $<P_n>$ is a Cauchy sequence. Indeed, observe that,

$$\delta(P_h,P_k) \leq e_\delta(p_h,p_k) + D(p_k)$$

and that

$$lim_{k\to\infty}D(p_k) = lim_{k\to\infty}f(pl(p_k)) = f(lim_{k\to\infty}\ pl(p_k)) = f(1) = 0.$$

Moreover, given $\varepsilon > 0$, let ε' be such that $f(\varepsilon') = \varepsilon$. Then $m\in N$ exists such that, $incl(p_h,p_k) = f^{[-1]}(e_\delta(p_h,p_k)) > \varepsilon'$ and therefore $e_\delta(p_h,p_k) \leq \varepsilon$ for any $h\geq m$, $k\geq m$. Let $P' \in M$ be the limit of the sequence $<P_n>$. Then, since

$$e_\delta(B_n(P'),p_n) \leq e_\delta(B_n(P'),P_n) \leq 1/n + \delta(P',P_n),$$

we have that $lim_{n\to\infty}e_\delta(B_n(P'),p_n) = 0$ and therefore that $i(P') = P$. □

7. Final remarks

As was proved by L. M. Blumenthal in [1], given an integer $n\in N$, it is possible to add to the theory of metric spaces MS a suitable set of axioms ES to obtain a theory $T = MS\cup ES$ for the Euclidean n-dimensional metric space. Obviously, the axioms in T refer to the points and the distance between points as primitives. Now, assume as primitives the regions and the graded inclusion between regions. Then, in account of the proposed definitions of point and distance between points, we can interpret the axioms in ES as properties of the regions and of the graded inclusion. Consequently, we can consider the theory $T^* = \{A1, A2, A3, A4, A5\}\cup ES$. The models of T^* are the graded inclusion spaces whose associated metric space (M,δ) is an Euclidean metric space. Then T^* gives a point-free approach to Euclidean geometry in which, in accordance with Whitehead's ideas, all the notions in the Euclidean geometry can be expressed in terms of regions and graded inclusion between regions. Observe that in accordance with Theorem 6.2 such a theory is consistent. In fact, a model of T^* is obtained from the class Re of the nonempty regular bounded subsets of (M,δ) and from any additive generator such that $f(0) = \infty$ (as an example the function $-log$). Obviously, the so obtained theory is not satisfactory. In fact, it should be more interesting to find some "natural" and more direct system of axioms for the Euclidean graded inclusion spaces of regions in which the characteristic properties of the regions in an Euclidean space are emphasized.

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