STRATIFIED OPERATORS AND GRADED CONSEQUENCE RELATIONS

by
Giangiacomo Gerla

(A more complete version of this technical note can be find in Chapter 7 of my book in Fuzzy Logic by Kluwer Editor)

1. GRADED DEDUCTIVE TOOLS FOR GRADED INFORMATION

In accordance with Tarski's point of view, we identify a crisp deductive apparatus with a *"deduction operator",* i.e. a closure operator $\mathcal{D} : P(\mathbb{F}) \to P(\mathbb{F})$ where $\mathbb{F}$ is the set of formulas in a logic. Given a set X of formulas, we interpret $\mathcal{D}(X)$ as the set of logical consequences of X. Now, we can imagine a *"stratified"* deduction apparatus, i.e. the availability of various deductive instruments each with a related degree of validity. We can represent such a state of affairs assuming that, for every $\lambda \in [0,1]$, a crisp deduction operator $\mathcal{D}_\lambda$ is defined. Given a set X of formulas, we interpret $\mathcal{D}_\lambda(X)$ as the set of formulas that we can derive from X by using arguments which are *"reliable"* to degree λ. More generally, it is possible that the available information and the deduction apparatus are both stratified. In this case, we represent the stratified information by a fuzzy set $v : \mathbb{F} \to [0,1]$. Then, if we denote by $C(v,\lambda)$ the closed cut $\{\alpha \in \mathbb{F} : v(\alpha) \geq \lambda\}$, in the case $\alpha \in \mathcal{D}_\lambda(C(v,\lambda))$ for a suitable $\lambda \in [0,1]$, we say that *α is a consequence of v to degree λ*. Obviously, we must consider the lower-constraint for the truth degree of α which is the best we are able to get. Then, it is natural to consider the number

$$\mathcal{D}(v)(\alpha) = Sup\{\lambda \in [0,1] : \alpha \in \mathcal{D}_\lambda(C(v,\lambda))\} \tag{1.1}$$

as the best lower-constraint for this truth degree. This suggests a way to define new fuzzy logics that will be useful to investigate the interesting notion of a graded consequence relation proposed in Chakraborty [1988].

In the sequel we denote by U the interval [0,1].

2. STRATIFIED FUZZY CLOSURE OPERATORS

The following definition enables us to associate any family of crisp operators with a fuzzy operator.

Definition 2.1. Let $(J_\lambda)_{\lambda\in U}$ be a family of operators in a set S and let J be the fuzzy operator defined by setting, for every $s \in \mathcal{F}(S)$ and $x \in S$,

$$J(s)(x) = Sup\{\lambda \in U : x \in J_\lambda(C(s,\lambda))\}. \tag{2.1}$$

Then we say that J is the *fuzzy operator associated with* $(J_\lambda)_{\lambda\in U}$.

We say that a family $(J_\lambda)_{\lambda\in U}$ of operators is a *chain* provided that $(J_\lambda(X))_{\lambda\in U}$ is a chain for every subset X, i.e.,

(i) J_0 is the map constantly equal to S,
(ii) $(J_\lambda)_{\lambda\in U}$ is order-reversing.
We say that $(J_\lambda)_{\lambda\in U}$ is a *continuous chain* provided that $(J_\lambda(X))_{\lambda\in U}$ is a continuous chain for every subset X, i.e.,
(j) J_0 is the map constantly equal to S,
(jj) $J_\mu(X) = \bigcap_{\lambda<\mu} J_\lambda(X)$ for every subset X and $\mu \in U$.

Definition 2.1 extends the notion of canonical extension of an operator proposed in Biacino L., Gerla G., [1996]. Indeed, it is easy to prove that the following proposition holds true.

Proposition 2.2. *Let J be associated with a chain $(J_\lambda)_{\lambda\in U}$ and H a crisp operator. Then the following are equivalent:*
(a) *J is an extension of H.*
(b) *$J_\lambda = H$ for every $0 < \lambda < 1$.*
(c) *J is the canonical extension of H.*

Of course, we are interested in families of closure operators.

Theorem 2.3. *Let $(J_\lambda)_{\lambda\in U}$ be a family of closure operators and let J be the associated operator. Then J is a fuzzy a-c-operator which is not, in general, a closure operator. Assume that $(J_\lambda)_{\lambda\in U}$ is a chain, then J is a fuzzy closure operator.*

Proof. Trivially, J is order-preserving. In order to prove the inclusion property, observe that, since $C(s,\lambda) \subseteq J_\lambda(C(s,\lambda))$, we have

$$s(x) = Sup\{\lambda \in U : x \in C(s,\lambda)\} \leq Sup\{\lambda \in U : x \in J_\lambda(C(s,\lambda))\} = J(s)(x).$$

To show an example in which J is not a closure operator, let $\boldsymbol{R}$ be the real line and, for every subset X of $\boldsymbol{R}$, denote the topological closure of X by $c(X)$ and the smallest closed convex subset containing X by $<X>$. Then we can consider the order-preserving family $(J_\lambda)_{\lambda\in U}$ defined by setting $J_\lambda(X) = c(X)$ for any $\lambda < 0.5$ and $J_\lambda(X) = <X>$ for $\lambda \geq 0.5$. We claim that the associated operator J is not a closure operator. Indeed, consider the fuzzy subset s defined by setting $s(x) = |x|$ if $-0.5 < x < 0.5$ and, otherwise, by setting $s(x) = 0$. Then, for every $\lambda \neq 0$, $C(s,\lambda) = (-0.5,\lambda] \cup [\lambda,0.5)$ if $\lambda < 0.5$ and $C(s,\lambda) = \emptyset$ if $\lambda \geq 0.5$. Consequently, since

$$J(s)(x) = Sup\{\lambda \in U : x \in J_\lambda(C(s,\lambda))\} = Sup\{\lambda < 0.5 : x \in c(C(s,\lambda))\},$$

it is $J(s)(x) = |x|$ if $-0.5 \leq x \leq 0.5$ and, otherwise, $J(s)(x) = 0$. On the other hand, since $J_{0.5}(C(J(s),0.5)) = <\{-0.5, +0.5\}> = [-0.5,+0.5]$, we have that $J(J(s))(x) = 0.5$ for every $x \in [-0.5,+0.5]$. This proves that $J(J(s)) \neq J(s)$.

Finally, assume that $(J_\lambda)_{\lambda\in U}$ is a chain. Then, to prove that $J(J(s)) = J(s)$, it is sufficient to prove that every cut $C(J(s),\lambda)$ is a fixed point for J_λ. Indeed, in this case

$$J(J(s))(x) = Sup\{\lambda \in U : x \in J_\lambda(C(J(s),\lambda))\}$$
$$= Sup\{\lambda \in U : x \in C(J(s),\lambda)\} = J(s)(x).$$

Now, observe that $(J_\lambda(C(s,\lambda)))_{\lambda\in U}$ is a chain of subsets of S. In fact, $J_0(C(s,0)) = S$ and if $\lambda \leq \lambda'$, then $J_\lambda(C(s,\lambda)) \supseteq J_\lambda(C(s,\lambda')) \supseteq J_{\lambda'}(C(s,\lambda'))$. Furthermore, observe that if $\mu \leq \lambda$, then every fixed point for J_μ is a fixed point for J_λ. In particular, $J_\mu(C(s,\mu))$ is a fixed point for J_λ. By recalling that the intersection of a class of fixed points is a fixed point and by observing that,

$$C(J(s),\lambda) = \bigcap_{\mu<\lambda} J_\mu(C(s,\mu)),$$

we conclude that $C(J(s),\lambda)$ is a fixed point for J_λ. □

The following proposition, whose proof we omit, shows that every fuzzy closure operator obtained by a chain of closure operators can be obtained by a continuous chain of closure operators.

Proposition 2.4. *Let $(J_\lambda)_{\lambda\in U}$ be any chain of closure operators and set, for every $\lambda \in U$ and $X \subseteq S$,*

$$J''_\lambda(X) = \bigcap_{\mu<\lambda} J_\mu(X). \tag{2.3}$$

Then $(J''_\lambda)_{\lambda\in U}$ is a continuous chain of closure operators. Moreover, the fuzzy closure operator associated with $(J_\lambda)_{\lambda\in U}$ coincides with the operator associated with $(J''_\lambda)_{\lambda\in U}$.

Definition 2.5. Let $(J_\lambda)_{\lambda\in U}$ be a family of closure operators and let J' be the associated fuzzy operator. We define the *fuzzy closure operator associated with* $(J_\lambda)_{\lambda\in U}$ as the closure operator J generated by J'. In this case we say that J is *stratified*. If $(J_\lambda)_{\lambda\in U}$ is a chain, we say that J is *well-stratified*.

An example of a fuzzy operator which is stratified but not well-stratified is given at the end of Section 4. In Castro, Trillas, Cubillo [1994] a a *fuzzy implication* is a fuzzy relation $Imp : S \times S \rightarrow U$ such that, for any x, y, z in S,

(a) $Imp(x,x) = 1$ (reflexivity),

(b) $Imp(x,y) \wedge Imp(y,z) \leq Imp(x,z)$ (transitivity)

The fuzzy operator J *associated with Imp* is defined by setting

$$J(s)(z) = Sup\{s(x) \wedge Imp(x,z) : x \in S\}. \tag{2.4}$$

We claim that J is well-stratified. Indeed, set, for any $\lambda \in U$,

$$J_\lambda(X) = \{x \in S : \exists x' \in X \text{ such that } Imp(x',x) \geq \lambda\}. \tag{2.5}$$

It is easy to demonstrate that $(J_\lambda)_{\lambda\in U}$ is a chain of closure operators and that J is the fuzzy closure operator J associated with this chain. Note that if Imp is not crisp, J is not an extension of a crisp operator and therefore, by Proposition 2.2, J is not a canonical extension of a crisp operator.

3. STRATIFIED CLOSURE SYSTEMS

Now, we define a notion of stratified closure system which is well related to the notion of stratified closure operator.

Definition 3.1. Let $(C_\lambda)_{\lambda\in U}$ be a family whose elements are classes of subsets of a given set S. Then the class of fuzzy subsets of S

$$C = \{s \in \mathcal{F}(S) : \mathrm{C}(s,\lambda) \in C_\lambda \text{ for every } \lambda \neq 0\} \quad (3.1)$$

is said to be *the class of fuzzy subsets associated with* $(C_\lambda)_{\lambda\in U}$.

Proposition 3.2. *Let C be the class of fuzzy subsets associated with the family* $(C_\lambda)_{\lambda\in U}$. *Then X is a crisp element of C iff, for any* $\lambda \neq 0$, *X is a crisp element of* C_λ. *In other words,* $\bigcap_{\lambda\neq 0} C_\lambda$ *is the class of crisp elements of C.*

Proof. If s is a crisp fuzzy set and $X = Supp(s)$, then $s \in C$ iff $X = \mathrm{C}(s,\lambda) \in C_\lambda$ for every $\lambda \neq 0$. □

Obviously, we are interested in families of closure systems, obviously. We say that a family of closure systems $(C_\lambda)_{\lambda\in U}$ is a *chain* if

(i) $C_0 = \{S\}$,

(ii) $(C_\lambda)_{\lambda\in U}$ is order-preserving.

Such a family is called a *continuous chain* if, for every $\lambda \in U$,

(j) $C_0 = \{S\}$,

(jj) $C_\lambda = Sup\{C_\mu : \mu < \lambda\}$.

Here the operator *Sup* is the join in the lattice of closure systems and hence (jj) means that C_λ is the closure system generated by $\bigcup_{\mu<\lambda} C_\mu$. This notion is well related to the notion of continuous chain for closure operators.

(3.1) generalizes the formula for the canonical extension of a class of subsets. More precisely:

Proposition 3.3. *Let C be the fuzzy system associated with a chain* $(C_\lambda)_{\lambda\in U}$ *of classes of subsets. Then C is the canonical extension of a crisp class* $\mathcal{H}$ *iff every* C_λ, *with* $\lambda \neq 0$, *coincides with* $\mathcal{H}$.

Proof. Suppose $C = \mathcal{H}^*$ where $\mathcal{H}$ is a class of subsets. Then, by Proposition 3.2, $\mathcal{H} = \bigcap_{\lambda\neq 0} C_\lambda$. Assume that $\lambda \neq 0$ exists such that $C_\lambda \neq \mathcal{H}$ and let X be an element of C_λ such that $X \notin \mathcal{H}$. Now, if $s = \lambda \vee X$, then $\mathrm{C}(s,\mu) = S$ if $\mu < \lambda$, otherwise $\mathrm{C}(s,\mu) = X$. As a consequence, since $(C_\lambda)_{\lambda\in U}$ is a chain, $\mathrm{C}(s,\mu) \in C_\mu$ for every $\mu \in U$. This demonstrates that $s \in C$ while it is obvious that $s \notin \mathcal{H}^*$. This contradiction proves that $C_\lambda = \mathcal{H}$. The converse implication is trivial. □

Theorem 3.4. *Let* $(C_\lambda)_{\lambda\in U}$ *be a family of closure systems. Then the class C associated with* $(C_\lambda)_{\lambda\in U}$ *is a fuzzy closure system.*

Proof. Let $(s_i)_{i\in I}$ be a family of elements of C. Then, since for every $\lambda \in U$,

$$\mathrm{C}(\bigcap_{i\in I} s_i,\lambda) = \bigcap_{i\in I} \mathrm{C}(s_i,\lambda) \in C_\lambda,$$

we have that $\bigcap_{i\in I} s_i \in C$ and this proves that C is a closure system. □

Note that, differently from the closure operators, a fuzzy closure system C associated with a family of closure systems $(C_\lambda)_{\lambda\in U}$ is always an extension of a classical closure system, namely the system $\bigcap_{\lambda\neq 0} C_\lambda$.

Definition 3.5. Let C be the fuzzy closure system associated with a family $(C_\lambda)_{\lambda\in U}$ of closure systems, then C is said to be *stratified*. If $(C_\lambda)_{\lambda\in U}$ is a chain, then we say that C is *well-stratified*.

Notice that the class of stratified closure systems is closed under finite and infinite intersections and therefore is a closure system.

Any family $(J_\lambda)_{\lambda\in U}$ of closure operators defines a corresponding family $(Cs(J_\lambda))_{\lambda\in U}$ of closure systems and any family $(C_\lambda)_{\lambda\in U}$ of closure systems defines a corresponding family $(Co(C_\lambda))_{\lambda\in U}$ of closure operators. Indeed, we have the following natural equivalences:

Proposition 3.6. *Let $(J_\lambda)_{\lambda\in U}$ be a family of closure operators. Then*

$$(J_\lambda)_{\lambda\in U} \text{ is a chain } \Leftrightarrow \ (Cs(J_\lambda))_{\lambda\in U} \text{ is a chain}$$
$$(J_\lambda)_{\lambda\in U} \text{ is a continuous chain } \Leftrightarrow \ (Cs(J_\lambda))_{\lambda\in U} \text{ is a continuous chain.}$$

Proof. The first equivalence is trivial. Assume that $(J_\lambda)_{\lambda\in U}$ is a continuous chain. We claim that $Cs(J_\mu)$ is the closure system generated by $\bigcup_{\lambda<\mu} Cs(J_\lambda)$. In fact, it is obvious that $Cs(J_\mu)$ is a closure system containing $\bigcup_{\lambda<\mu} Cs(J_\lambda)$. Let C be a closure system containing the class $\bigcup_{\lambda<\mu} Cs(J_\lambda)$ and let X be an element of $Cs(J_\mu)$. Then, since $X = J_\mu(X) = \bigcap_{\lambda<\mu} J_\lambda(X)$ and $J_\lambda(X)$ is an element of $Cs(J_\lambda)$, X is an intersection of elements of $\bigcup_{\lambda<\mu} Cs(J_\lambda)$. Hence $X \in C$ and this proves that $C \supseteq Cs(J_\mu)$.

Suppose $(Cs(J_\lambda))_{\lambda\in U}$ is a continuous chain. Then $(J_\lambda)_{\lambda\in U}$ is a chain and $Cs(J_\mu)$ is the closure system generated by $\bigcup_{\lambda<\mu} Cs(J_\lambda)$. Consequently, since $J_\mu(X)$ is an element of $Cs(J_\mu)$, a family $(X_{\lambda(i)})_{i\in I}$ of subsets of S exists such that $\lambda(i) < \mu$, $X_{\lambda(i)}$ is a fixed point of $J_{\lambda(i)}$ and $J_\mu(X) = \bigcap_{i\in I} X_{\lambda(i)}$. Furthermore, since $X_{\lambda(i)} \supseteq X$ entails $X_{\lambda(i)} \supseteq J_{\lambda(i)}(X)$, we have

$$J_\mu(X) = \bigcap_{i\in I} J_{\lambda(i)}(X) \supseteq \bigcap_{\lambda<\mu} J_\lambda(X).$$

Now, $(J_\lambda)_{\lambda\in U}$ is order-reversing and therefore $J_\mu(X) \subseteq \bigcap_{\lambda<\mu} J_\lambda(X)$. Thus, $J_\mu(X) = \bigcap_{\lambda<\mu} J_\lambda(X)$ and this proves the continuity of $(J_\lambda)_{\lambda\in U}$. □

The next theorem says that Definitions 2.1 and 3.1 are related in a natural way:

Theorem 3.7. *Let J be the closure operator associated with a family $(J_\lambda)_{\lambda\in U}$ of closure operators and C the closure system associated with the family $(Cs(J_\lambda))_{\lambda\in U}$ of closure systems. Then, $J = Co(C)$, that is*

$$\begin{array}{ccc} (J_\lambda)_{\lambda\in U} & \longrightarrow & (Cs(J_\lambda))_{\lambda\in U} \\ \downarrow & & \downarrow \end{array}$$

$$J \longleftarrow C.$$

Proof. Let J' be the operator associated with $(J_\lambda)_{\lambda\in U}$. In order to prove $J = Co(C)$ we can prove that $Cs(J) = C$, i.e., that $C = Cs(J')$. Let s be an element of C, then every cut $C(s,\lambda)$ belongs to $Cs(J_\lambda)$ and therefore it is a fixed point for J_λ. Then

$$J'(s)(x) = Sup\{\lambda \in U : x \in J_\lambda(C(s,\lambda))\} = Sup\{\lambda \in U : x \in C(s,\lambda)\} = s(x)$$

and this proves that $s \in Cs(J')$. Conversely, if $J'(s) = s$, then, for every $x \in S$,

$$Sup\{\lambda\in U : x \in J_\lambda(C(s,\lambda))\} = s(x).$$

In other words, $x \in J_\lambda(C(s,\lambda))$ implies $\lambda \le s(x)$ and this proves that $x \in C(s,\lambda)$. Then, since $J_\lambda(C(s,\lambda))$ is contained in $C(s,\lambda)$, $C(s,\lambda)$ is a fixed point for J_λ. Thus s is an element of C. □

In a similar way one demonstrates the next theorem:

Theorem 3.8. *Let C be the fuzzy closure system associated with a family $(C_\lambda)_{\lambda\in U}$ of closure systems and J the closure operator associated with the family $(Co(C_\lambda))_{\lambda\in U}$ of closure operators. Then, $C = Cs(J)$, that is*

$$\begin{array}{ccc} (C_\lambda)_{\lambda\in U} & \longrightarrow & (Co(C_\lambda))_{\lambda\in U} \\ \downarrow & & \downarrow \\ C & \longleftarrow & J. \end{array}$$

Corollary 3.9. *If J is a closure operator, then*

J stratified $\Leftrightarrow$ $Cs(J)$ stratified,

J well-stratified $\Leftrightarrow$ $Cs(J)$ well-stratified.

If C is a fuzzy closure system, then

C stratified $\Leftrightarrow$ $Co(C)$ stratified,

C well-stratified $\Leftrightarrow$ $Co(C)$ well-stratified.

Proposition 3.10. *Assume that J_1 and J_2 are two (well) stratified closure operators. Then the closure operator generated by the product $J_1 \circ J_2$ is (well) stratified.*

Proof. Observe that $Cs(J_1)$ and $Cs(J_2)$ are both (well) stratified and therefore $Cs(J_1)\cap Cs(J_2)$ is also (well) stratified. Moreover, it is easy to prove that $Cs(J_1)\cap Cs(J_2)$ is the set of fixed points of $J_1 \circ J_2$. This completes the proof. □

4. A CHARACTERIZATION OF STRATIFIED CLOSURE SYSTEMS

Observe that every fuzzy closure system determines a family of closure systems in a natural way.

Definition 4.1. Let C be a class of fuzzy subsets and set, for every $\lambda \in U$

$$\mathcal{H}(C,\lambda) = \{\mathrm{C}(s,\lambda) : s \in C\}. \tag{4.1}$$

Then we say that $(\mathcal{H}(C,\lambda))_{\lambda \in U}$ is *the family of classes associated with C*.

Proposition 4.2. *For every fuzzy closure system C, and $\lambda \in U$, the class $\mathcal{H}(C,\lambda)$ is a closure system. Moreover, if C is associated with a family $(C_\lambda)_{\lambda\in U}$ of closure systems then, for every $\lambda \in U$,*

$$\mathcal{H}(C,\lambda) \subseteq C_\lambda.$$

Finally, in the case that $(C_\lambda)_{\lambda\in U}$ is a continuous chain,

$$\mathcal{H}(C,\lambda) = C_\lambda.$$

Proof. Let $(X_i)_{i\in I}$ be a family of elements of $\mathcal{H}(C,\lambda)$. Then a family $(s_i)_{i\in I}$ of elements of C exists such that $X_i = \mathrm{C}(s_i,\lambda)$. Since $\bigcap_{i\in I}X_i = \mathrm{C}(\bigcap_{i\in I}s_i,\lambda)$ and $\bigcap_{i\in I}s_i$ belongs to C, the set $\bigcap_{i\in I}X_i$ belongs to $\mathcal{H}(C,\lambda)$. This demonstrates that $\mathcal{H}(C,\lambda)$ is a closure system.

Assume that C is associated with a family $(C_\lambda)_{\lambda\in U}$ of closure systems and let $X \in \mathcal{H}(C,\lambda)$. Then an element s_λ in C exists such that $\mathrm{C}(s_\lambda,\lambda) = X$. The fact that every λ-cut of an element in C belongs to C_λ enables us to conclude that $X \in C_\lambda$.

Assume that $(C_\lambda)_{\lambda\in U}$ is a continuous chain and that $X \in C_\lambda$. Then, since C_λ is the closure operator generated by $\cup_{\mu<\lambda} C_\mu$, $X = \cap_{i\in I}X_i$ where each X_i is an element of a suitable $C_{\mu(i)}$, for $\mu(i) < \lambda$. Consider the fuzzy subset $s_i = X_i \vee \mu(i)$. We claim that $s_i \in C$. Indeed, since S belongs to any closure system, in the case $t \leq \mu(i)$ we have $\mathrm{C}(s_i,t) = S \in C_t$. In the case $t > \mu(i)$, since $\mathrm{C}(s,t) = X_i$ and $X_i \in C_{\mu(i)}$, from the inclusion $C_{\mu(i)} \subseteq C_t$, it follows that $\mathrm{C}(s,t) \in C_t$. From $s_i \in C$ it follows that $X_i = \mathrm{C}(s_i,\lambda) \in \mathcal{H}(C,\lambda)$. Since $\mathcal{H}(C,\lambda)$ is a closure system, this demonstrates that $X = \cap_{i\in I}X_i \in \mathcal{H}(C,\lambda)$. □

Definition 4.3. Given a fuzzy closure system C, we denote by C^* the fuzzy closure system associated with the family $(\mathcal{H}(C,\lambda))_{\lambda\in U}$ and we say that C^* is *the stratified closure system associated with C*.

In other words, we set

$$C^* = \{s \in \mathcal{F}(S) : \mathrm{C}(s,\lambda) \in \mathcal{H}(C,\lambda) \text{ for every } \lambda \in U\}. \tag{4.2}$$

In a series of papers we defined the canonical extension of a crisp closure system C as the fuzzy closure systems

$$C^* = \{s : \mathrm{C}(s,\lambda) \in C \text{ for any } \lambda\in U \}$$

The following proposition, whose proof is trivial, shows that the notation in Definition 4.3 is coherent with such a definition.

Proposition 4.4. *If C is a classical closure system, then the stratified closure system associated with C by* (4.2) *coincides with the canonical extension of C.*

The following proposition shows that the map associating any fuzzy closure system C with the fuzzy closure system C^* defines a closure operator in the lattice of fuzzy closure systems. Furthermore, it characterizes the stratified fuzzy closure systems as the fixed points of such an operator.

Theorem 4.5. *Assume that* C, C_1 *and* C_2 *are fuzzy closure systems. Then*
(i) $C \subseteq C^*$,
(ii) $C_1 \subseteq C_2 \Rightarrow C_1^* \subseteq C_2^*$,
(iii) $(C^*)^* = C^*$.
Moreover,
$$C = C^* \Leftrightarrow C \text{ is a stratified closure system.}$$

Proof. Properties (i), (ii) and (iii) are obvious. It is self-evident that if $C = C^*$, then C is stratified. Assume that C is the stratified closure system associated with a family $(C_\lambda)_{\lambda\in U}$ of closure systems. Then from the inclusion $\mathcal{H}(C,\lambda) \subseteq C_\lambda$ we have that $C^* \subseteq C$ and therefore that $C^* = C$.
The converse implication is trivial. □

Let $\mathcal{T}$ be a class of fuzzy subsets. Then we define $Q(\mathcal{T})$ by setting
$$Q(\mathcal{T}) = \{\lambda \vee \mathrm{C}(s,\lambda) : s \in \mathcal{T}, \lambda \in U\}. \quad (4.3)$$

Proposition 4.6. For every fuzzy closure system C,
$$C \subseteq C^* \subseteq c(Q(C))$$
where $c(Q(C))$ is the fuzzy closure system generated by $Q(C)$. Consequently,
$$C \supseteq Q(C) \Rightarrow C = C^* = c(Q(C)).$$
Furthermore, if $(\mathcal{H}(C,\lambda))_{\lambda\in U}$ is a chain,
$$C^* = c(Q(C))$$
and therefore,
$$C = c(Q(C)) \Leftrightarrow C = C^*.$$

Proof. Let $s \in C^*$ and observe that
$C^* = \{s \in \mathcal{F}(S) : \text{for every } \lambda \in U, \mathrm{C}(s,\lambda) = \mathrm{C}(s_\lambda,\lambda) \text{ for a suitable } s_\lambda \in C\}$.
Then,
$$s = \bigcap_{\lambda\in U} \lambda \vee \mathrm{C}(s,\lambda) = \bigcap_{\lambda\in U} \lambda \vee \mathrm{C}(s_\lambda,\lambda),$$
and therefore that $s \in c(Q(C))$. This proves that $C^* \subseteq c(Q(C))$.

Suppose that $(\mathcal{H}(C,\lambda))_{\lambda\in U}$ is a chain, and let $s \in C$. Then, since $\mathrm{C}(s,\lambda) \in \mathcal{H}(C,\lambda) \subseteq \mathcal{H}(C,\mu)$ for every $\mu > \lambda$ and

$$\mathrm{C}(\lambda \vee \mathrm{C}(s,\lambda), \mu) = \begin{cases} S & \text{if } \mu \le \lambda, \\ \mathrm{C}(s,\lambda) & \text{if } \mu > \lambda, \end{cases}$$

we may conclude that $Q(C) \subseteq C^*$ and therefore, that $c(Q(C)) \subseteq C^*$. □

Examples. Let E be a finite dimensional Euclidean space, CS the class of closed subsets of E and CCS the class of closed convex subsets of S. Then we obtain a continuous chain $(C_\lambda)_{\lambda\in U}$ by setting $C_0 = \{S\}$ and

$$C_\lambda = \begin{cases} CCS & \text{if } 0 < \lambda \leq 0.5, \\ CS & \text{otherwise.} \end{cases}$$

Denote the well-stratified closure system associated with this family by C. Then

$s \in C \Leftrightarrow \mathrm{C}(s,\lambda)$ is closed for $\lambda > 0.5$ and closed and convex for $\lambda \leq 0.5$.

Also, we have

$$C = C^* = c(Q(C)) \text{ and } \mathcal{H}(C,\lambda) = C_\lambda$$

and the class of crisp elements of C coincides with CCS.

A counterexample can be achieved by exchanging CCS with CS in defining $(C_\lambda)_{\lambda\in U}$, i.e., by setting

$$C_\lambda = \begin{cases} CS & \text{if } 0 < \lambda \leq 0.5, \\ CCS & \text{otherwise.} \end{cases}$$

Then the obtained family is order-reversing. Moreover, if C is the fuzzy closure system associated with $(C_\lambda)_{\lambda\in U}$,

$s \in C \Leftrightarrow \mathrm{C}(s,\lambda)$ is closed for $\lambda \leq 0.5$ and closed and convex for $\lambda > 0.5$.

Trivially, since C is stratified, $C = C^*$. We claim that

$$Q(C) \nsubseteq C. \tag{4.4}$$

Indeed, let X and Y be two disjoint closed subsets such that $X \cup Y$ is not convex and Y is convex and define s by

$$s(x) = \begin{cases} 1 & \text{if } x \in Y, \\ 0.5 & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $s \in C$ and that $\lambda \vee \mathrm{C}(s,\lambda) = \lambda \vee (X \cup Y)$ for every $\lambda \leq 0.5$. Now, since $\mathrm{C}(\lambda \vee X \cup Y,\mu) = X \cup Y$ for every $\mu > \lambda$, we have that $\mathrm{C}(\lambda \vee X \cup Y,\mu) \notin C_\mu$ for every $\mu > 0.5$. Thus, $\lambda \vee \mathrm{C}(s,\lambda) \notin C$ and this proves that $Q(C)$ is not contained in C. As a consequence of (4.4) we have

$$C^* \neq c(Q(C)). \tag{4.5}$$

Another interesting property is that, for any $\lambda \in U$,

$$C_\lambda = \mathcal{H}(C,\lambda). \tag{4.6}$$

In fact, let X be an element of C_λ. Then in the case $\lambda > 0.5$ the set X is closed and convex. Hence $X \in C$ and $X \in \mathcal{H}(C,\lambda)$. If $\lambda \leq 0.5$, then $\lambda \wedge X$ is an element of C such that $X = \mathrm{C}(\lambda \wedge X,\lambda)$. This proves that $X \in \mathcal{H}(C,\lambda)$.

Finally, note that C is not well-stratified. Indeed, otherwise, let $(Q_\lambda)_{\lambda\in U}$ be a continuous chain of closure systems whose associate system is C. Then by

Proposition 4.2, we have that $Q_\lambda = \mathcal{H}(C,\lambda) = C_\lambda$. This is absurd. In fact, $(C_\lambda)_{\lambda\in U}$, is not order-preserving.

Co(*C*) is an example of fuzzy operator which is stratified and not well-stratified.

5. A CHARACTERIZATION OF STRATIFIED OPERATORS

Given a fuzzy closure operator *J*, we may define a family $(\mathcal{K}(J,\lambda))_{\lambda\in U}$ of classical operators by setting

$$\mathcal{K}(J,\lambda)(X) = \mathrm{C}(J(\lambda\wedge X),\lambda) \tag{5.1}$$

for every $\lambda \in U$. If *J* is a deduction operator of a fuzzy logic, then we can interpret $\mathcal{K}(J,\lambda)(X)$ as the set of formulas which are consequences at least to degree λ of the formulas in *X* assumed at least to degree λ.

The following proposition shows that $\mathcal{K}$ and $\mathcal{H}$ are related in accordance with the diagrams:

$$\begin{array}{ccc} J & \longrightarrow & Cs(J) \\ \downarrow & & \downarrow \\ K(J,\lambda) & \longleftarrow & \mathcal{H}(Cs(J),\lambda) \end{array} \qquad \begin{array}{ccc} C & \longrightarrow & Co(C) \\ \downarrow & & \downarrow \\ \mathcal{H}(C,\lambda) & \longleftarrow & K(Co(C),\lambda). \end{array}$$

Proposition 5.1 (Castro [1993]) *Given a fuzzy closure operator J*, $(\mathcal{K}(J,\lambda))_{\lambda\in U}$ *is a family of closure operators. More specifically, we have*

$$\mathcal{K}(J,\lambda) = Co(\mathcal{H}(Cs(J),\lambda)). \tag{5.2}$$

Given a fuzzy closure system C, we have

$$\mathcal{H}(C,\lambda) = Cs(\mathcal{K}(Co(C),\lambda)). \tag{5.3}$$

Proof. Let *X* be a subset of *S* and suppose $x \in Co(\mathcal{H}(Cs(J),\lambda))(X)$. Then, since $\mathcal{H}(Cs(J),\lambda)$ is the class of the λ-cuts of $Cs(J)$, $x \in \mathrm{C}(s,\lambda)$ for every $s \in Cs(J)$ such that $X \subseteq \mathrm{C}(s,\lambda)$. Taking $s = J(\lambda \wedge X)$, since $s \in Cs(J)$ and

$$X = \mathrm{C}(\lambda \wedge X,\lambda) \subseteq \mathrm{C}(J(\lambda \wedge X),\lambda) = \mathrm{C}(s,\lambda),$$

we have $x \in \mathrm{C}(J(\lambda \wedge X),\lambda) = \mathcal{K}(J,\lambda)(X)$.

Conversely, suppose $x \in \mathcal{K}(J,\lambda)(X)$. Then $J(\lambda \wedge X)(x) \geq \lambda$ and hence, for any $s \in Cs(J)$ such that $s \supseteq \lambda \wedge X$, we have $x \in \mathrm{C}(s,\lambda)$. Thus, since $s \supseteq \lambda\wedge X$ iff $\mathrm{C}(s,\lambda) \supseteq X$, for every $s \in Cs(J)$ such that $\mathrm{C}(s,\lambda) \supseteq X$, we have $x \in \mathrm{C}(s,\lambda)$. This proves that $x \in Co(\mathcal{H}(Cs(J),\lambda))(X)$.

In order to prove (5.3), we apply (5.2) to the fuzzy closure operator *Co*(*C*) by obtaining

$$\mathcal{K}(Co(C),\lambda) = Co(\mathcal{H}(C,\lambda)).$$

This equation is equivalent to (5.3). □

Definition 5.2. Given a fuzzy closure operator J we denote by J^* the fuzzy closure operator associated with the family $(\mathcal{K}(J,\lambda))_{\lambda\in U}$ of closure operators and we say that J^* is *the stratified operator associated with J*.

The above notation is in accordance with the notation for the canonical extension of a classical closure operator. Indeed, given an operator J, denote by J' the fuzzy operator defined by setting, for every fuzzy subset s,

$$J'(s) = J(Supp(s)).$$

Then, the following proposition holds:

Proposition 5.3. *Let J be a classical closure operator. Then the stratified operator associated with J' coincides with the canonical extension of J.*

Proof. Observe that

$$Sup\{\lambda \in U : x \in J(\mathrm{C}(s,\lambda))\} = Sup\{\lambda \in U : x \in J(Supp(\lambda\wedge\mathrm{C}(s,\lambda),\lambda))\}$$
$$= Sup\{\lambda \in U : J'(\lambda \wedge \mathrm{C}(s,\lambda))(x) \geq \lambda\}.$$

This demonstrates both that the *a*-*c*-closure operator associated with the family $(\mathcal{K}(J',\lambda))_{\lambda\in U}$ of closure operators is a closure operator and that this operator coincides with the canonical extension. □

The next proposition shows that the notion of stratified closure system C^* associated with a fuzzy closure system C is strictly related to the notion of stratified fuzzy operator J^* associated with a fuzzy closure operator J. Indeed, the following diagrams commute:

$$\begin{array}{ccc} C & \longrightarrow & Co(C) \\ \downarrow & & \downarrow \\ C^* & \longleftarrow & Co(C)^* \end{array} \qquad\qquad \begin{array}{ccc} J & \longrightarrow & Cs(J) \\ \downarrow & & \downarrow \\ J^* & \longleftarrow & Cs(J)^* \end{array}$$

Proposition 5.4. *Let C and J be a fuzzy closure system and a fuzzy closure operator, respectively. Then*

$$C^* = Cs(Co(C)^*) \ ; \ J^* = Co(Cs(J)^*). \tag{5.4}$$

Proof. If J is a fuzzy closure operator, then by Proposition 3.6, $J^* = Co(C)$ where C is the closure system associated with the family $(Cs(\mathcal{K}(J,\lambda)))_{\lambda\in U}$. Moreover, since by Proposition 5.1 $\mathcal{K}(J,\lambda) = Co(\mathcal{H}(Cs(J),\lambda))$, we have that $Cs(\mathcal{K}(J,\lambda)) = \mathcal{H}(Cs(J),\lambda)$. This proves the first part of the proposition.

Let C be a fuzzy closure system and set $J = Co(C)$. Then from the proved equality we obtain that $(Co(C))^* = Co(Cs(Co(C))^*) = Co(C^*)$ and therefore $Cs(Co(C)^*) = Cs(Co(C^*)) = C^*$. □

The following Theorem shows that Definition 5.2 gives an interior operator in the lattice of fuzzy closure operators. Furthermore, it characterizes the stratified closure operators as the fixed points of such an operator.

Theorem 5.5. *For every fuzzy closure operator J,*

$$J^* \leq J \quad ; \quad J_1 \leq J_2 \Rightarrow J_1^* \leq J_2^* \quad ; \quad J^{**} = J^*.$$

Moreover,

$$J = J^* \Leftrightarrow J \text{ is a stratified closure operator.}$$

Proof. Inclusion $Cs(J)^* \supseteq Cs(J)$ entails that $J^* = Co(Cs(J)^*) \leq Co(Cs(J)) = J$. Likewise, since $Cs(J^*) = Cs(J)^*$,

$$J^{**} = Co(Cs(J^*)^*) = Co(Cs(J)^*) = J^*.$$

Moreover, from $J_1 \leq J_2$ it follows that $C_1 \supseteq C_2$ and, hence, that $C_1^* \supseteq C_2^*$. So, $J_1^* = Co((Cs(J_1))^*) \leq Co(Cs(J_2)^*) = J_2^*$. Finally, $J = J^*$ iff $J = Co((Cs(J))^*)$ iff $Cs(J) = Cs(J)^*$ iff a family $(C_\lambda)_{\lambda \in U}$ exists such that $Cs(J)$ is associated with it. In turn, this happens iff J is associated with $(Co(C_\lambda))_{\lambda\in U}$. □

6. STRATIFIED DEDUCTION SYSTEMS

We can interpret the definitions and the results in the previous sections in terms of deduction systems. Indeed, we interpret a family $((\mathcal{F},\mathcal{D}_\lambda))_{\lambda\in U}$ of deduction systems as a deduction apparatus stratified in accordance with the reliability of the deductive instruments used. More precisely, given a set X of formulas, we interpret $\mathcal{D}_\lambda(X)$ as the set of formulas we can derive from X to degree λ.

Definition 6.1. Let $((\mathcal{F},\mathcal{D}_\lambda))_{\lambda\in U}$ be a family of crisp deduction systems and $\mathcal{D}$ the closure operator associated with $(\mathcal{D}_\lambda)_{\lambda\in U}$. Then $(\mathcal{F},\mathcal{D})$ is called, *the fuzzy deduction system associated with* $((\mathcal{F},\mathcal{D}_\lambda))_{\lambda\in U}$. In this case, we say that $(\mathcal{F},\mathcal{D})$ is *stratified*. If $((\mathcal{F},\mathcal{D}_\lambda))_{\lambda\in U}$ is a chain, we say that $(\mathcal{F},\mathcal{D})$ is *well-stratified*.

It is rather natural to admit that $(\mathcal{D}_\lambda)_{\lambda\in U}$ is a continuous chain. Indeed, given a set X of formulas, condition $\mathcal{D}_0(X) = \mathcal{F}$ means that every formula can be considered as a consequence of X (at least) to degree zero. The inclusion $J_\mu(X) \subseteq \bigcap_{\lambda<\mu} J_\lambda(X)$, means that, for every $\lambda < \mu$, if x is a consequence of X (at least) to degree μ, then x is a consequence of X (at least) to degree λ. Condition $J_\mu(X) \supseteq \bigcap_{\lambda<\mu} J_\lambda(X)$ says that if x is a consequence of X (at least) to degree λ for any $\lambda < \mu$, then x is a consequence of X (at least) to degree μ.

If $(\mathcal{D}_\lambda)_{\lambda\in U}$ is a chain, $\mathcal{D}$ can be defined in a more expressive way by the relation $\vdash_\lambda$ defined by setting $v \vdash_\lambda \alpha$ everywhere $\alpha \in \mathcal{D}_\lambda(\mathrm{C}(v,\lambda))$. Indeed,

$$\mathcal{D}(v)(\alpha) = Sup\{\lambda \in U : v \vdash_\lambda \alpha\}. \qquad (6.1)$$

Trivially, if *Tau* is the fuzzy subset of tautologies of $\mathcal{D}$ and Tau_λ the set of tautologies of $\mathcal{D}_\lambda$, $Tau(\alpha) = Sup\{\lambda \in U : \alpha \in Tau_\lambda\}$ or, equivalently,

$$Tau = \cup_{\lambda \in U}(\lambda \wedge Tau_\lambda). \tag{6.2}$$

In particular, α is a tautology of $(\mathbb{F},\mathcal{D})$ iff α is a tautology of all the deduction systems $(\mathbb{F},\mathcal{D}_\lambda)$.

Theorem 6.2. *Let* $(\mathbb{F},\mathcal{D})$ *be a stratified fuzzy deduction system. Then, for every set* X *of formulas and* $\mu \in U$,

$$\mathcal{D}(\mu \wedge X) = (\mu \wedge \mathcal{D}(X)) \cup Tau. \tag{6.3}$$

Moreover,

$$Inc(\mathcal{D}) = 1. \tag{6.4}$$

Proof. To prove (6.3) assume that $\mathcal{D}$ is associated with a family of $(\mathcal{D}_\lambda)_{\lambda \in U}$ of closure operators. Since

$$C(\mu \wedge X, \lambda) = \begin{cases} X & \text{if } \lambda \leq \mu \\ \varnothing & \text{if } \lambda > \mu, \end{cases}$$

we have that

$$\mathcal{D}(\mu \wedge X)(x) = Sup\{\lambda \in U : x \in \mathcal{D}_\lambda(X),\ \lambda \leq \mu\} \vee Sup\{\lambda \in U : x \in \mathcal{D}_\lambda(\varnothing),\ \lambda > \mu\}.$$

Assume that $x \in \mathcal{D}_\lambda(\varnothing)$ for a suitable $\lambda > \mu$. Then, since

$$Sup\{\lambda \in U : x \in \mathcal{D}_\lambda(\varnothing),\ \lambda > \mu\} > \mu \geq Sup\{\lambda \in U : x \in \mathcal{D}_\lambda(X),\ \lambda \leq \mu\}$$

it is

$$\mu \wedge \mathcal{D}(X)(x) = Sup\{\lambda \in U : x \in \mathcal{D}_\lambda(X),\ \lambda \leq \mu\}$$

Because $\mathcal{D}$ is order-preserving,

$$\mathcal{D}(\varnothing)(x) \leq \mathcal{D}(\mu \wedge X)(x) = Sup\{\lambda \in U : x \in \mathcal{D}_\lambda(\varnothing),\ \lambda > \mu\} \leq \mathcal{D}(\varnothing)(x)$$

and $\mathcal{D}(\mu \wedge X)(x) = \mathcal{D}(\varnothing)(x)$. Then

$$\mathcal{D}(\varnothing)(x) = ((\mu \wedge \mathcal{D}(X)(x)) \vee \mathcal{D}(\varnothing))(x).$$

Assume that, for every $\lambda > \mu$, $x \notin \mathcal{D}_\lambda(\varnothing)$. Then $Sup\{\lambda \in U : x \in \mathcal{D}_\lambda(\varnothing),\ \lambda > \mu\} = 0$ and $\mathcal{D}(\varnothing)(x) \leq \mu$. As a consequence

$$\mathcal{D}(\mu \wedge X)(x) = Sup\{\lambda \in U : x \in \mathcal{D}_\lambda(X),\ \lambda \leq \mu\} = \mu \wedge \mathcal{D}(X)(x)$$
$$= ((\mu \wedge \mathcal{D}(X)) \vee \mathcal{D}(\varnothing))(x).$$

To prove (6.4), set X equal to the whole set of formulas $\mathbb{F}$. Then (6.3) becomes

$$\mathcal{D}(s^\mu) = s^\mu \cup Tau.$$

Consequently, $Inc(s^\mu) \geq \mu$ and therefore, $Inc(\mathcal{D}) \geq Sup\{Inc(s^\mu) : \mu \in U\} = 1$. □

The proof of the following theorem is trivial.

Theorem 6.3. *Let* $(\mathbb{F},\mathcal{D})$ *be a fuzzy deduction system. Then* $(\mathbb{F},\mathcal{D})$ *is stratified iff the class of its theories is a stratified closure system. Moreover, if* $(\mathbb{F},\mathcal{D})$ *is associated with the family* $((\mathbb{F},\mathcal{D}_\lambda))_{\lambda\in U}$ *of deduction systems, then*

$$\tau \text{ is a theory of } (\mathbb{F},\mathcal{D}) \Leftrightarrow \text{ every cut } C(\tau,\lambda) \text{ of } \tau \text{ is a theory of } (\mathbb{F},\mathcal{D}_\lambda).$$

Finally,

$$X \text{ is a crisp theory of } (\mathbb{F},\mathcal{D}) \Leftrightarrow X \text{ is a theory of any } (\mathbb{F},\mathcal{D}_\lambda).$$

Proposition 4.6 and Theorem 4.5 entail the next theorem:

Theorem 6.4. *Assume that, for every* $\lambda \in U$,

$$\tau \text{ theory of } (\mathbb{F},\mathcal{D}) \Rightarrow \lambda \vee C(\tau,\lambda) \text{ theory of } (\mathbb{F},\mathcal{D}).$$

Then $(\mathbb{F},\mathcal{D})$ *is stratified.*

By recalling that the deduction operator of a canonical similarity logic is generated by the product of two stratified operators, from Proposition 3.10 we obtain:

Theorem 6.5. *Any canonical similarity logic is well-stratified.*

7. SEQUENTS AND CONSEQUENCE RELATIONS

We call *sequent* any element of the set *SEQ*Š= $\mathcal{P}(\mathbb{F}) \times \mathbb{F}$Ā, i.e., any pair (X,α) where X is a set of formulas and α a formula. A sequent (X,α) represents the metalogical claim that α is a consequence of the set X of formulas. In this section we begin by giving the basic notions of the theory of crisp consequence relations. We call a *conclusion relation* any set of sequents, i.e., any binary relation $\vdash$ from $\mathcal{P}(\mathbb{F})$ to $\mathbb{F}$. Given $X \in \mathcal{P}(\mathbb{F})$ and $\alpha \in \mathbb{F}$, we write $X \vdash \alpha$ to denote that $(X,\alpha) \in \vdash$. Given $Z \in \mathcal{P}(\mathbb{F})$, we write $X \vdash Z$ to denote that $X \vdash \alpha$ for any formula α in Z.

Definition 7.1. A conclusion relation $\vdash$ is a *consequence relation* if

(i) $X \vdash \alpha$ whenever $\alpha \in X$,
(ii) $X \vdash \alpha \Rightarrow X \cup Y \vdash \alpha$,
(iii) $X \vdash Z$ and $X \cup Z \vdash \alpha \Rightarrow X \vdash \alpha$.

If $\vdash$ is a consequence relation and $X \vdash \alpha$, then we say that α *is a consequence of* X. The meanings of the above conditions are apparent. Condition (i) says that every formula in X is a consequence of X, condition (ii) that the logic under consideration is monotone, (iii) that if the set of formulas Z follows from X and we are able to prove α from $X \cup Z$, then we may prove α directly from X.

There is a strict connection between the operators and the conclusion relations.

Definition 7.2. Given an operator J, we define $\vdash_J$ by setting

$$X \vdash_J \alpha \iff \alpha \in J(X). \tag{7.1}$$

Given a conclusion relation $\vdash$ we denote by $J_\vdash$ the operator defined by

$$J_\vdash(X) = \{\alpha \in \mathbb{F} : X \vdash \alpha\}. \tag{7.2}$$

These definitions enable us to define a bijective correspondence between the class of operators and the class of conclusion relations.

Proposition 7.3. *Let J be an operator and let $\vdash$ be its associated conclusion relation. Then*

$$J_\vdash = J. \tag{7.3}$$

Let $\vdash$ be a conclusion relation and denote its associated operator by J. Then

$$\vdash_J = \vdash. \tag{7.4}$$

Proof. For every $X \subseteq \mathbb{F}$,

$$J_\vdash(X) = \{\alpha \in \mathbb{F} : X \vdash_J \alpha\} = \{\alpha \in \mathbb{F} : \alpha \in J(X)\} = J(X).$$

Likewise, for every $X \subseteq \mathbb{F}$ and $\alpha \in \mathbb{F}$

$$X \vdash_J \alpha \iff \alpha \in J_\vdash(X) \iff X \vdash \alpha. \qquad \square$$

Definitions (7.1) and (7.2) establish also a one-one correspondence between the crisp consequence relations and the closure operators.

Theorem 7.4. *Let $\vdash$ be a conclusion relation. Then*

$\vdash$ is a consequence relation $\iff$ $J_\vdash$ is a closure operator.

Let $J : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be an operator. Then

J is a closure operator $\iff$ $\vdash_J$ is a consequence relation.

Proof. Assume that $\vdash$ is a consequence relation. Then, from (i) it follows that $J_\vdash(X) \supseteq X$ and from (ii) that $X \supseteq Y$ implies $J_\vdash(X) \supseteq J_\vdash(Y)$. In order to prove that $J_\vdash(J_\vdash(X)) = J_\vdash(X)$, observe that, since $X \vdash \beta$ for every $\beta \in J_\vdash(X)$, by (iii),

$$J_\vdash(X) \vdash \alpha \Rightarrow X \vdash \alpha.$$

Thus, $J_\vdash(J_\vdash(X)) = \{\alpha \in \mathbb{F} : J_\vdash(X) \vdash \alpha\} \subseteq J_\vdash(X)$ and $J_\vdash(J_\vdash(X)) = J_\vdash(X)$.

Assume that J is a closure operator. Then, the inclusion property $J(X) \supseteq X$ entails that $X \vdash_J \alpha$ for every $\alpha \in X$, and this demonstrates (i). In order to prove (ii), suppose $X \vdash_J \alpha$ and $Y \supseteq X$. Then, since $\alpha \in J(X)$ and J is order-preserving, $\alpha \in J(Y)$, i.e., $Y \vdash_J \alpha$. In order to prove (iii), assume that $X \vdash_J \beta$ for every $\beta \in Z$ and that $X \cup Z \vdash_J \alpha$. Then, $Z \subseteq J(X)$ and $\alpha \in J(X \cup Z)$. Consequently, since $X \cup Z \subseteq J(X)$ and $\alpha \in J(X \cup Z) \subseteq J(J(X)) = J(X)$, we have that that X $\vdash_J \alpha$. Thus $\vdash_J$ is a consequence relation.

The remaining part of the proposition follows from Proposition 7.3. $\square$

For instance if J is the identity map, then the corresponding relation $\vdash_J$ is the usual membership relation. As a matter of fact, by (i) of Definition 7.1, this relation is the least crisp consequence relation.

The next theorem shows that the theory of consequence relations coincides with the theory of the abstract deduction systems.

Theorem 7.5. *A conclusion relation* $\vdash$ *is a consequence relation iff a deduction system* ($\mathbb{F}$,$\mathcal{D}$) *exists such that*

$$X \vdash \alpha \Leftrightarrow \alpha \in \mathcal{D}(X). \qquad (7.5)$$

Proof. Let $\vdash$ be a consequence relation and let $\mathcal{D} = J_{\vdash}$. Then $\mathcal{D}$ is a closure operator satisfying (7.5). Conversely, if ($\mathbb{F}$,$\mathcal{D}$) is a deduction system satisfying (7.5), then $\vdash$ coincides with $\vdash_{\mathcal{D}}$. So, $\vdash$ is a consequence relation. □

8. GRADED CONSEQUENCES AND SEQUENT CALCULUS

The concept of graded consequence relation was proposed in Chakraborty [1988] as a graded extension of the crisp abstract concept of consequence relation $\vdash$. We call a *graded conclusion relation* any fuzzy subset of sequents, i.e., any fuzzy relation $g : SEQ \rightarrow U$ from $\mathcal{P}(\mathbb{F})$ to $\mathbb{F}$. If X is a set of formulas and α a formula, we write $g(X \vdash \alpha)$ instead of $g(X,\alpha)$. Moreover, given another set Z of formulas, we set

$$g(X \vdash Z) = Inf\{g(X \vdash z) : z \in Z\}. \qquad (8.1)$$

Definition 8.1. We say that a graded conclusion relation g is a *graded consequence relation* if, for every X, Y, $Z \in \mathcal{P}(\mathbb{F})$ and $\alpha \in \mathbb{F}$,

(i) $g(X \vdash \alpha) = 1$ for every $\alpha \in X$,
(ii) $g(X \cup Y \vdash \alpha) \geq g(X \vdash \alpha)$,
(iii) $g(X \vdash \alpha) \geq g(X \vdash Z) \wedge g(X \cup Z \vdash \alpha)$.

If $\lambda = g(X \vdash \alpha)$ we say that *α is a consequence of X at least to degree λ.* The following proposition, whose proof is trivial, summarizes some elementary properties of the graded consequences.

Proposition 8.2. *Let g be a graded consequence. Then, for* X, Y, Z, X_1, X_2, Y_1, Y_2 *subsets of* $\mathbb{F}$ *and* $(Y_i)_{i\in I}$ *family of subsets of* $\mathbb{F}$,

(i) $g(X \vdash X) = 1$,
(ii) $g(X \cup Y \vdash Z) \geq g'(X \vdash Z)$,
(iii) $g(X \vdash Y) \geq g(X \vdash Z) \wedge g(X \cup Z \vdash Y)$,
(iv) $g(X \vdash \bigcup_{i\in I} Y_i) = Inf\{g(X \vdash Y_i) : i \in I\}$,
(v) $X_1 \subseteq X_2 \Rightarrow g(X_1 \vdash Y) \leq g(X_2 \vdash Y)$,
(vi) $Y_1 \subseteq Y_2 \Rightarrow g(X \vdash Y_1) \geq g(X \vdash Y_2)$,
(vii) $X \supseteq Y \Rightarrow g(X \vdash Y) = 1$,
(viii) $g(X \vdash Y) \geq g(X \vdash Z) \wedge g(Z \vdash Y)$.

It is possible to interpret the crisp consequence relations as theories of a suitable *H*-system with an infinitary inference rule. In fact, first we consider a crisp *H*-system $\mathcal{S} = (\mathbb{A},\mathbb{R})$ which we call *minimal sequent calculus*, such that

- the set of formulas is the set *SEQ* of sequents,
- the set $\mathbb{A}$ of logical axioms is $\{(X,x) : x \in X\}$,
- there is a finitary rule:

$$\frac{(X, \alpha)}{(X \cup Y, \alpha)}$$

and an infinitary rule:

$$\frac{\{(X, \beta) : \beta \in Z\}\ ,\ (X \cup Z, \alpha)}{(X , \alpha)} \quad .$$

Proposition 8.3. *The class of theories of the minimal sequent calculus coincides with the class of crisp consequence relations.*

Proof. Indeed, with reference to Definition 7.1, a set $\vdash$ of sequents contains the set of logical axioms iff it satisfies (i). $\vdash$ is closed under the finitary rule iff it satisfies (ii), $\vdash$ is closed under the infinitary rule iff it satisfies (iii). □

The next proposition shows that the class of graded consequence relations is the canonical extension of the class of crisp consequence relations.

Proposition 8.4. *The following are equivalent:*

(a) $g : SEQ \to U$ *is a graded consequence relation.*

(b) *every cut* $C(g,\lambda)$ *is a consequence relation.*

In other words, the class of graded consequence relations is the canonical extension of the class of consequence relations.

Proof. (a) $\Rightarrow$ (b) It is self-evident that $C(g,\lambda)$ satisfies (i) and (ii) of Definition 7.1. In order to prove (iii), suppose $(X \cup Z,\alpha) \in C(g,\lambda)$ and $(X,z) \in C(g,\lambda)$ for every $z \in Z$. Then $g(X \cup Z \vdash \alpha) \geq \lambda$ and $g(X \vdash z) \geq \lambda$ for every $z \in Z$. Consequently, $g(X \vdash Z) \geq \lambda$ and by (iii) of Definition 8.1 this implies that $g(X \vdash \alpha) \geq \lambda$. Hence, $(X,\alpha) \in C(g,\lambda)$.

(b) $\Rightarrow$ (a) Let X be a set of formulas and $x \in X$. Then, the fact that $C(g,1)$ is a consequence relation entails that $(X,x) \in C(g,1)$, i.e., $g(X \vdash x) = 1$. Let Y be a set of formulas containing X, and $\lambda = g(X \vdash \alpha)$. Then, since $(X,\alpha) \in C(g,\lambda)$ and $Y \supseteq X$, we have $(Y,\alpha) \in C(g,\lambda)$, i.e., $g(Y \vdash \alpha) \geq \lambda = g(X \vdash \alpha)$. Finally, given any set Z of formulas, set

$$\lambda = Inf(\{g(X \vdash z) : z \in Z\}) \wedge g(X \cup Z \vdash \alpha).$$

Then, since $C(g,\lambda)$ is a consequence relation, $(X,z) \in C(g,\lambda)$ for every $z \in Z$ and $(X \cup Z, \alpha) \in C(g,\lambda)$, we may conclude that $(X,\alpha) \in C(g,\lambda)$. Thus

$$g(X \vdash \alpha) \geq Inf(\{g(X \vdash z) : z \in Z\}) \wedge g(X \cup Z \vdash \alpha). \quad □$$

We indicate the canonical extension of the minimal sequent calculus by $\mathcal{S}^*$, where how to extend an infinitary rule in a fuzzy infinitary rule is evident. Then

- the fuzzy set of logical axioms is

$$\{((X,x) : x \in X\},$$

- we have a finitary rule

$$\frac{(X,\alpha)}{(X \cup Y)} \quad ; \quad \frac{\lambda}{\lambda}$$

- we have an infinitary rule

$$\frac{\{(X,\beta) : \beta \in Z\},\ (X \cup Z,\alpha)}{(X,\ \alpha)} \quad ; \quad \frac{S,\ \lambda}{Inf(S) \wedge \lambda}$$

The following theorem holds:

Theorem 8.5. *The class of graded consequence relations coincides with the class of theories of the canonical extension* $\mathcal{S}^*$ *of the minimal sequent calculus* $\mathcal{S}$.

Proof. This is trivial. □

In particular, the class of graded consequence relations is a closure system and any fuzzy conclusion relation can be extended to a consequence relation (see Castro, Trillas, Cubillo [1994]).

9. FINITE SEQUENT CALCULUS AND COMPACT GRADED CONSEQUENCES

It is possible to avoid the infinitary inference rules provided that we only consider compact graded consequences. In the following, if X is a set we denote the class of finite subsets of X by $\mathcal{P}_f(X)$. A conclusion relation $\vdash$ is *compact*, if

$$X \vdash \alpha \Leftrightarrow \text{ there exists } X_f \in \mathcal{P}_f(X) \text{ such that } X_f \vdash \alpha.$$

Trivially, if $\vdash$ is a conclusion relation, then

$$\vdash \text{ is compact } \Leftrightarrow J_{\vdash} \text{ is compact,}$$

if J is an operator, then

$$J \text{ is compact } \Leftrightarrow \vdash_J \text{ is compact.}$$

A graded conclusion relation g is said to be *compact* if

$$g(X,\alpha) = Sup\{g(X_f \vdash \alpha) : X_f \in \mathcal{P}_f(X)\}. \quad (9.1)$$

Proposition 9.1. *A graded conclusion g is a compact graded consequence iff it satisfies* (9.1) *and*

(i) $g(X \vdash \alpha) = 1$ *for every* $\alpha \in X$,
(ii) $g(X \cup Y \vdash \alpha) \geq g(X \vdash \alpha)$,
(iii) $g(X \vdash \alpha) \geq g(X \vdash z) \wedge g(X \cup \{z\} \vdash \alpha)$.

Proof. Let g be is a compact graded consequence. Then it is evident that (9.1), (i), (ii) and (iii) are satisfied. Conversely, assume these conditions are satisfied. Then, first we demonstrate that, for every finite set $Z_f = \{z_1,...,z_n\}$,

$$g(X \vdash \alpha) \geq (Inf\{g(X \vdash z) : z \in Z_f\}) \wedge g(X \cup Z_f \vdash \alpha).$$

Indeed, such an inequality coincides with (iii) for $n = 1$. Moreover, by induction hypothesis,

$$\begin{aligned} g(X \vdash \alpha) &\geq (Inf\{g(X \vdash z) : z \in \{z_1,...,z_{n-1}\}\}) \wedge g(X \cup \{z_1,...,z_{n-1}\} \vdash \alpha) \\ &\geq (Inf\{g(X \vdash z) : z \in \{z_1,...,z_{n-1}\}\})) \wedge g(X \cup \{z_1,...,z_{n-1}\} \vdash z_n) \\ &\quad \wedge g(X \cup \{z_1,...,z_{n-1},z_n\} \vdash \alpha) \\ &\geq (Inf\{g(X \vdash z) : z \in \{z_1,...,z_{n-1}\}\})) \wedge g(X \vdash z_n) \wedge g(X \cup Z_f \vdash \alpha) \\ &= (Inf\{g(X \vdash z) : z \in Z_f\}) \wedge g(X \cup Z_f \vdash \alpha). \end{aligned}$$

Let Z be any set, then,

$$\begin{aligned} &(Inf\{g(X \vdash z) : z \in Z\}) \wedge g(X \cup Z \vdash \alpha) \\ &= (Inf\{g(X \vdash z) : z \in Z\}) \wedge (Sup\{g(X \cup Z_f \vdash \alpha) : Z_f \in \mathcal{P}_f(Z)\} \\ &= Sup\{(Inf\{g(X \vdash z) : z \in Z\}) \wedge g(X \cup Z_f \vdash \alpha) : Z_f \in \mathcal{P}_f(Z)\} \\ &\leq Sup\{Inf(g(X \vdash z) : z \in Z_f\}) \wedge g(X \cup Z_f \vdash \alpha) : Z_f \in \mathcal{P}_f(Z)\} \leq g(X \vdash \alpha). \end{aligned}$$

□

We call *finite sequent* any sequent (X,α) in which X is finite and we denote the set of finite sequents by SEQ_f. If g is compact, then g is completely defined by its restriction to SEQ_f. Conversely, let h be a fuzzy subset of SEQ_f and set

$$g(X,\alpha) = Sup\{h(X_f,\alpha) : X_f \text{ is a finite subset of } X\}.$$

Then g is a compact graded conclusion relation. If h satisfies (ii), then g is an extension of h we indicate as *the compact extension of h*.

Proposition 9.2. *g is a compact graded consequence relation iff g is the compact extension of a fuzzy relation* $h : \mathcal{P}_f(\mathbb{F}) \times \mathbb{F} \to U$ *satisfying*

(j) $h(X,\alpha) = 1$ *for every* $\alpha \in X$,
(jj) $h(X \cup Y, \alpha) \geq h(X,\alpha)$,
(jjj) $h(X,\alpha) \geq h(X,\beta) \wedge h(X \cup \{\beta\},\alpha)$.

Proof. If g is a compact conclusion relation, then it is obvious that its restriction h to SEQ_f satisfies (j), (jj) and (jjj). Conversely, let g be the compact extension of a fuzzy relation h satisfying (j), (jj) and (jjj). Then, by using Proposition 9.1 we can prove g is a graded consequence relation by proving g satisfies (i) and (ii) and

$$g(X,\alpha) \geq g(X,\beta) \wedge g(X \cup \{\beta\}, \alpha). \tag{9.3}$$

Now, (i) and (ii) are trivial. In order to prove (9.3) observe that

$$g(X,\alpha) = Sup\{h(X_f,\alpha) : X_f \in \mathcal{P}_f(X)\} \geq Sup\{h(X_f,\beta) \wedge h(X_f \cup \{\beta\},\alpha) : X_f \in \mathcal{P}_f(X)\}.$$

On the other hand,

$g(X,\beta) \wedge g(X \cup \{\beta\},\alpha)$
$= (Sup\{h(X_1,\beta)) : X_1 \in \mathcal{P}_f(X)\}) \wedge (Sup\{h(X_2 \cup \{\beta\},\alpha) : X_2 \in \mathcal{P}_f(X)\})$
$=Sup\{h(X_1,\beta) \wedge h(X_2 \cup \{\beta\},\alpha) : X_1, X_2 \in \mathcal{P}_f(X)\}$.

Now, observe that, by setting $X_f = X_1 \cup X_2$,
$h(X_f,B) \wedge h(X_f \cup \beta,\alpha) \geq h(X_1,\beta) \wedge h(X_2 \cup \{\beta\},\alpha)$.
Then, we can conclude that $g(X,\alpha) \geq g(X,\beta) \wedge g(X \cup \{\beta\},\alpha)$. □

Proposition 9.2 enables us to relate the compact graded consequence relations with the theories of the canonical extension of a suitable sequent calculus. In fact, let $\mathcal{S}_f = (\mathbb{A},\mathbb{R})$ be the H-system such that

- SEQ_f is the set of formulas,
- the set $\mathbb{A}$ of logical axioms is $\{(X,x) \in SEQ_f : x \in X\}$,
- there are the following rules:

$$\frac{(Y,\alpha)}{(X \cup Y,\ \alpha)} \quad ; \quad \frac{(X,\beta),\ (X \cup \{\beta\},\ \alpha)}{(X,\ \alpha)} \quad .$$

We call *minimal finite-sequent calculus* such a system. Then, we have the following theorem:

Theorem 9.3. *Let $\mathcal{S}_f^*$ be the canonical extension of $\mathcal{S}_f$. Then, g is a compact graded consequence relation iff g is (the compact extension of) a theory of $\mathcal{S}_f^*$.*

Proof. It is evident that $h : SEQ_f \to U$ satisfies (j), (jj) and (jjj) of Proposition 9.2 iff h is a theory of $\mathcal{S}_f^*$. □

10. GRADED CONSEQUENCES AND STRATIFIED OPERATORS

The following theorem shows that we can identify the graded consequence relations with the continuous chains of consequence relations:

Theorem 10.1. *A conclusion relation g is a graded consequence relation iff a continuous family $(\vdash_\lambda)_{\lambda \in U}$ of consequence relations exists such that*

$$g(X \vdash \alpha) = Sup\{\lambda \in U : X \vdash_\lambda \alpha\}. \qquad (10.1)$$

Proof. Given $\lambda \in U$, denote the conclusion relation $C(g,\lambda)$ by $\vdash_\lambda$. Then
$$g(X \vdash \alpha) = Sup\{\lambda \in U : (X,\alpha) \in C(g,\lambda)\} = Sup\{\lambda \in U : X \vdash_\lambda \alpha\}.$$
So, the proof follows from Proposition 8.4. □

In Theorem 7.5 we observed that a conclusion relation is a consequence relation iff a closure operator J exists such that
$$X \vdash \alpha \Leftrightarrow \alpha \in J(X).$$

The question arises whether such a connection holds also for the graded consequence relations and the fuzzy closure operators.

Proposition 10.2. *Let J be a fuzzy closure operator and define the graded conclusion relation g by setting*

$$g(X \vdash \alpha) = J(X)(\alpha) \qquad (10.2)$$

for every $X \subseteq \mathbb{F}$ and $\alpha \in \mathbb{F}$. Then, in general, *g satisfies* (i) *and* (ii) *but not* (iii).

Proof. A straightforward verification proves the first part of the proposition. In the following example (iii) is not satisfied (M. K. Chakraborty, personal communication). Let $\mathbb{F} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and let s_1 and s_2 be the two fuzzy subsets of $\mathbb{F}$ defined by setting

$$s_1(\alpha_1) = s_1(\alpha_3) = 1 \ , \ s_1(\alpha_2) = 0.7 \ , \ s_1(\alpha_4) = 0.8$$

and

$$s_2(\alpha_1) = s_2(\alpha_3) = s_2(\alpha_4) = 1 \ , \ s_2(\alpha_2) = 0.9 \ .$$

Then, the class $C = \{s_1, s_2\}$ defines a fuzzy closure operator J. Namely, for every fuzzy subset s and $\alpha \in \mathbb{F}$,

$$J(s)(\alpha) = Inf\{s_i(\alpha) : s_i \supseteq s\}.$$

Take $X = \{\alpha_1, \alpha_3\}$ and $Z = \{\alpha_4\}$. Then, a simple calculation gives

$$J(X)(\alpha_2) = 0.7 \ , \ J(X \cup Z)(\alpha_2) = 0.9 \ , \ J(X)(\alpha_4) = 0.8.$$

So, if g is the conclusion relation associated with J,

$$g(X \vdash \alpha_2) = 0.7, \ \ Inf\{g(X \vdash z) : z \in Z\} = 0.8, \text{ and } g(X \cup Z \vdash \alpha_2) = 0.9.$$

Hence,

$$g(X \vdash \alpha_2) < (Inf\{g(X \vdash z) : z \in Z\}) \wedge g(X \cup Z \vdash \alpha_2).$$

This demonstrates that (iii) is not satisfied. □

The following theorem shows that we can extend Theorem 7.5 to the graded consequences provided we confine ourselves to the well-stratified deduction systems (Gerla [1996]).

Theorem 10.3. *A fuzzy conclusion relation $g : \mathcal{P}(\mathbb{F}) \times \mathbb{F} \to U$ is a graded consequence relation iff a well-stratified deduction system $(\mathbb{F}, \mathcal{D})$ exists such that*

$$g(X \vdash \alpha) = \mathcal{D}(X)(\alpha) \qquad (10.3)$$

for every X subset of $\mathbb{F}$ and $\alpha \in \mathbb{F}$.

Proof. Let g be a graded consequence relation and, for every $\lambda \in U$, let $\mathcal{D}_\lambda$ be the deduction operator associated with the consequence relation $C(g,\lambda)$, that is

$$\mathcal{D}_\lambda(X) = \{x \in \mathbb{F} : g(X \vdash x) \geq \lambda\}.$$

Moreover, denote by $\mathcal{D}$ the closure operator associated with $(\mathcal{D}_\lambda)_{\lambda \in U}$. Then

$$g(X \vdash \alpha) = Sup\{\lambda \in U : g(X \vdash \alpha) \geq \lambda\} = Sup\{\lambda \in U : \alpha \in \mathcal{D}_\lambda(X)\} = \mathcal{D}(X)(\alpha).$$

So, we must prove only that $(\mathcal{D}_\lambda)_{\lambda\in U}$ is a continuous chain. Let X be a set of formulas. Then, trivially, $\mathcal{D}_0(X) = \mathbb{F}$. Furthermore, if $\mu \in U$, then

$$x \in \mathcal{D}_\mu(X) \Leftrightarrow g(X \vdash x) \geq \mu \Leftrightarrow g(X \vdash x) \geq \lambda \text{ for every } \lambda < \mu$$
$$\Leftrightarrow x \in \bigcap_{\lambda<\mu}\mathcal{D}_\lambda(X).$$

Conversely, let $(\mathbb{F},\mathcal{D})$ be the fuzzy deduction system associated with a given continuous chain $(\mathcal{D}_\lambda)_{\lambda\in U}$ of deduction systems and, for every $\lambda \in U$, denote by $\vdash_\lambda$ the consequence relation associated with $\mathcal{D}_\lambda$, that is $\vdash_\lambda = \{(X,x) : x \in \mathcal{D}_\lambda(X)\}$. We claim that $(\vdash_\lambda)_{\lambda\in U}$ is a continuous family. Indeed, $\vdash_0 = SEQ$ and

$$(X,x) \in \vdash_\mu \Leftrightarrow x \in \mathcal{D}_\mu(X) \Leftrightarrow x \in \mathcal{D}_\lambda(X) \text{ for every } \lambda < \mu$$
$$\Leftrightarrow (X,x) \in \vdash_\lambda \text{ for every } \lambda < \mu.$$

Thus, by Theorem 10.1 the conclusion relation g defined by (10.3) is a graded consequence relation. □

Theorem 10.3 enables us to find examples of graded consequence in a simple way. For instance, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two different deductive systems on the same set $\mathbb{F}$ of formulas and let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the related deduction operators. Moreover, assume that $\mathcal{S}_2$ is more powerful than $\mathcal{S}_1$, that is $\mathcal{D}_1(X) \subseteq \mathcal{D}_2(X)$ for every set X of formulas but that, at the same time, $\mathcal{S}_2$ is less reliable than $\mathcal{S}_1$. Then, a continuous family of closure operators is achieved by setting $J_\lambda = \mathcal{D}_2$ for $\lambda \leq 0.5$ and $J_\lambda = \mathcal{D}_1$ for $\lambda > 0.5$. We obtain the corresponding graded consequence relation g by

$$g(X \vdash x) = \begin{cases} 1 & \text{if } x \in \mathcal{D}_1(X), \\ 0.5 & \text{if } x \in \mathcal{D}_2(X) - \mathcal{D}_1(X), \\ 0 & \text{otherwise.} \end{cases} \tag{10.4}$$

Further examples of graded consequences are furnished by the canonical similarity logics. In fact, the deduction operators of these logics are well-stratified.

Theorem 10.4. *Let $Con : \mathcal{P}(\mathbb{F}) \times \mathbb{F} \to U$ the fuzzy relation associated with a canonical similarity logic. Then, Con is a graded consequence relation.*

Proof. See Theorem 6.5. □

Remark. Theorem 10.3 suggests a natural way to extend a graded consequence relation g in a fuzzy relation g^e from the lattice $\mathcal{F}(\mathbb{F})$ to $\mathbb{F}$. Indeed, it is sufficient to consider the stratified deduction operator $\mathcal{D}$ associated with g and to set

$$g^e(s \vdash \alpha) = \mathcal{D}(s)(\alpha). \tag{10.5}$$

Equivalently, we can set

$$g^e(s \vdash \alpha) = Sup\{\lambda \in U : g(C(s,\lambda) \vdash \alpha) \geq \lambda\} \quad (10.6)$$

for every $s \in \mathcal{F}(\mathbb{F})$ and $\alpha \in \mathbb{F}$. This suggests to examine the possibility of extending the definition of graded consequence by calling *fuzzy sequent* any element of the set $\underline{SEQ} = \mathcal{F}(\mathbb{F})\times\mathbb{F}$ and defining the graded consequence relations as suitable fuzzy subsets $Con : \underline{SEQ} \to U$ of $\underline{SEQ}$.

REFERENCES

Biacino L. [1993], Generated envelopes, *J. Math. Anal. Appl.*, **172**, 179-190.

Biacino L., Gerla G. [1984], Closure systems and L-subalgebras, *Inform. Sci.*, **32**, 181-195.

Biacino L., Gerla G. [1992]b, Generated necessities and possibilities, *Internat. J. of Intell. Systems*, **7**, 445-454.

Biacino L., Gerla G., [1993], Closure operators for fuzzy subsets, *EUFIT '93, First-European Congress on Fuzzy and Intelligent Technologies*, Aachen, 1441-1447.

Biacino L., Gerla G., [1996], An extension principle for closure operators, *J. of Math. Anal. Appl.,* **198**, 1-24.

Biacino L., Gerla G., [1998], Logics with approximate premises, *International J. of Intell. Syst.*, **13**, 1-10.

Biacino L., Gerla G., [1999], Closure Operators in Fuzzy Set Theory, in *Fuzzy Sets in Approximate Reasoning and Information Systems,* Edited by J. C. Bezdek, D. Dubois, H. Prade, Kluwer Academic Publishers, Dordrecht, 243-278.

Biacino L., Gerla G., [2000]a, Axiomatizability, continuity and computability for multivalued logics, unpublished paper.

Biacino L., Gerla G., [2000]b, Boolean logic and capacities, unpublished paper.

Biacino L., Gerla G., Ying M. [2000], Approximate reasoning based on similarity, *Math. Log. Quart.*, **46**, 77-86.

Birkhoff G. [1948], *Lattice Theory*, Amer. Math. Soc., New York.

Bison P., Chemello G., Sossai C., Trainito G., [1995], An Algebraic Logic for Autonomous Systems, *Intelligent Autonomous Systems*, IAS-4, Karlsruhe, 182-189.

Brown D. J., Suszko R. [1973], Abstract Logics, *Dissertationes Mathematicae*, **102**, 9-42.

Castro J.L. [1994], Fuzzy logics as families of bivaluated logics, *Fuzzy Sets and Systems*, **64**, 321-332.

Castro J.L., Trillas E. [1993], The management of the inconsistency in expert systems, *Fuzzy Sets and Systems*, **58**, 51-57.

Castro J.L., Trillas E., Cubillo S. [1994], On consequences in approximate reasoning, *J. of Applied Non-classical Logics*, **4**, 91-103.

Chang C.C., Keisler H.J. [1966], *Continuous Model Theory*, Princeton, Princeton University Press.

Chang C.C., Keisler H.J. [1973], *Model Theory*, North-Holland, Amsterdam.

Chakraborty M.K. [1988], Use of fuzzy set theory in introducing graded consequence in multiple valued logic, in *Fuzzy Logic in Knowledge-Based Systems, Decision and Control*, M.M. Gupta and T.Yamakawa (eds), North-Holland, Amsterdam, 247-257.

Chakraborty M.K. [1994], Graded consequence: further studies, *J. of Applied non-classical Logic*, **5**, 227-237.

Cleave J.P. [1974], The notion of logical consequence in the logic of inexact predicates, *Zeitschr. f. math. Logik und Grundlagen d. Math.*, **20**, 307-324.

Cohn P. M. [1965], *Universal Algebra*, Harper and Row, New York.

Di Nola A. , Gerla G. [1986], Fuzzy models of first order languages, *Zeitschr. f. math. Logik und Grundlagen d. Math.*, **32**, 331-340.

Di Nola A., Gerla G. [1987], Lattice valued algebras, *Stochastica*, **11**, 137-150.

Dubois D., Esteva F., Garcia P., Godo L., Prade H. [1996], *A logical approach to interpolation based on similarity relations*, Report de Recerca IIIA, Campus de la Universitat Autonoma de Barcelona.

Dubois D., Lang J., Prade H. [1986], Automated reasoning using possibilistic logic: semantics, belief revision and variable certainty weights. *Proc. 5th Workshop on Uncertainty in Artificial Intelligence*, Windsor, Ont., 81-87.

Dubois D., Lang J., Prade H. [1991], Fuzzy sets in approximate reasoning, part 2: logical approaches, *Fuzzy Sets and Systems,* **40**, 203-244.

Dubois D., Lang J., Prade H. [1994], Possibilistic logic, In: D.Gabbay, C.Hogger and J.Robinson (eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming*, vol. 3. Clarendon Press.

Dubois D., Prade H. [1987], Necessity measures and the resolution principle, *IEEE Trans. Systems Man and Cybernetics*, **17**, 474-478.

Dubois D., Prade H. [1988], *Possibility Theory*, Plenum Press, New York and London.

Dubois D., Prade H. [1990]b, Reasoning with inconsistent information in a possibilistic setting, *Proc. of the Europ. Conf. on A.I. (ECAI-90)*, Stockholm, 259-261.

Dubois D., Prade H. [1992], Combination of fuzzy information in the framework of possibility theory, In: Abidi and Gonzales (eds.), *Data Fusion in Robotics and Machine Intelligence*, Academic Press, 481-505.

Dubois D., Prade H. [1994], A survey of belief revision and updating rules in various uncertainty models, *Internat. J. of Intell. Systems*, **9**, 61-100.

Gerla G. [1985], Pavelka's fuzzy logic and free L-subsemigroups, *Zeitschr. f. math. Logik und Grundlagen d. Math.*, **31**, 123-129.

Gerla G. [1986], The category of the fuzzy models and Lowenheim-Skolem theorem, In *Mathematics of Fuzzy Systems*, Di Nola and Ventre (eds), Verlag TUV Rheinland, Germany, 121-141.

Gerla G. [1994]a, Comparing fuzzy and crisp deduction systems, *Fuzzy Sets and Systems*, **67**, 317-328.

Gerla G. [1994]d, An extension principle for fuzzy logics, *Mathematical Logic Quarterly*, **40**, 357-80.

Gerla G. [1996], Graded consequence relations and closure operators, *J. of Applied non-classical Logic*, **6**, 369-379.

Gerla G., [1999], Closure operators, fuzzy logic and constraints, in *Fuzzy Sests, Logics and Reasoning About Knowledge*, (D. Dubois, H. Prade E.P. Klement, Eds), Kluwer Academic Publishers, Dordrecht, 101-120.

Gerla G., Scarpati L. [1998], Extension principles for fuzzy set theory, *J. of Inform. Sci.*, **106**, 49-69.

Goguen J.A. [1968/69], The logic of inexact concepts, *Synthese*, **19**, 325-373.

Gottwald S. [1993], *Fuzzy Sets and Fuzzy Logic*, Wiesbaden: Vieweg.

Grätzer G. [1968], *Universal Algebra*, D. Van Nostrand Company, Toronto.

Hähnle R. [1993], *Automated Deduction in Multiple-valued Logics*, Clarendon Press, Oxford.

Hájek P. [1993], On logics of approximate reasoning, *Neural Network Word* 6/1993, 733-744.

Hájek P. [1998], *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht.

Hájek P. [1999]a, On the metamathematics of fuzzy logic, in *Discovering World with Fuzzy Logic: Perspectives and Approaches to Formalization of Human-Consistent Logical Systems*, (V. Novák, I. Perfilieva, editors), Springer-Verlag, Heidelberg.

Hájek P. [1999]b, Ten questions and one problem on fuzzy logic, *Annals of Pure and Applied Logic*, **96**, 157-165.

Höhle U. [1991], Foundations of fuzzy sets, *Fuzzy Sets and Systems,* **40**, 257-296.

Höhle U. [1996], On the Fundamentals of Fuzzy Set Theory, *J. Math. Anal. Appl.,* **201**, 786-826.

Horn A., Tarski A. [1948], Measures in Boolean algebras, *Trans. Am. Math. Soc.*, **64**, 467-497.

Kerre E.E. [1980], Fuzzy Sierpinski space and its generalizations, *J.Math. Anal. Appl.*, **74**, 318-324.

Lu J. J., Murray N. V., Rosenthal E. [1998], Signed Formulas and Fuzzy Operator Logics, *J. Automat. Reason.* **21**, 39-67.

Łukasiewicz J. [1970], *Selected works*, Studies in Logic and Foundations of Mathematics, North-Holland Publ.Comp., Amsterdam and PWN, Warszawa.

Mamdani, E.H., Gaines, B.R. eds. [1981], *Fuzzy Reasoning and its Applications*, Academic Press, London.

McNaughton R. [1951], A theorem about infinite-valued sentential logic, *J. of Symbolic Logic*, **16**, 1-13.

Mendelson E. [1964], *Introduction to Mathematical Logic*, D. Van Nostrand Company, Princeton.

Mundici D. [1992], Normal forms in infinite-valued logic: the case of one variable, *Lecture Notes in Computer Science*, **629**, 272-277.

Murali V. [1991], Lattice of fuzzy subalgebras and closure systems in I^X, *Fuzzy Sets and Systems*, **41**, 101-111.

Murali V. [1993], L-fuzzy Subalgebras in *Advances in Fuzzy Theory and Technology*, vol. 1, Ed. P.P. Wand, Bookwrights Press, Durham, North Carolina, 77-90.

Murray N. V., Rosenthal E. [1991], Resolution and path dissolution in multiple-valued logics, *Proc. of the 6th International Synposium on Methodologies for Intelligent Systems,* Springer vol. 542, 570-579.

Negoita C.V., Ralescu D.A. [1975]a, Representation theorems for fuzzy concepts, *Kybernetes*, **4**, 169-174.

Negoita C.V., Ralescu D.A. [1975]b, *Applications of Fuzzy Sets to Systems Analysis*, Birkhauser Verlag, Basel.

Novák V. [1987], First-Order fuzzy logic, *Studia Logica*, **46**, 87-109.

Novák V., [1990], On the syntactico-semantical completeness of first-order fuzzy logic I, *Kybernetika*, **26**, 47-66.

Novák V., [1990], On the syntactico-semantical completeness of first-order fuzzy logic II, Kybernetika, **26**, 134-152.

Novák V., Perfilieva I, Mockor J. [1999], *Mathematical Principles of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht.

Pavelka J. [1979]a, On fuzzy logic I: Many-valued rules of inference, *Zeitschr. f. math. Logik und Grundlagen d. Math.*, **25**, 45-52.

Pavelka J. [1979]b, On fuzzy logic II: Enriched residuated lattices and semantics of propositional calculi, *Zeitschr. f. math. Logik und Grundlagen d. Math.*, **25**, 119-134.

Pavelka J. [1979]c, On fuzzy logic III: Semantical completeness of some many-valued propositional calculi, *Zeitschr. f. math. Logik und Grundlagen d. Math.*, **25**, 447-464.

Pawlak Z. [1982], Rough sets, *International J.of Information and Computer Science*, **11**, 341-356.

Ramik J. [1983], Extension principle and fuzzy-mathematical programming, *Kybernetika*, **19**, 513-25.

Rogers H. [1967], *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York.

Ruspini E. [1991], On the semantics of fuzzy logic, *Int. J. of Approximate Reasoning*, **5**, 45-88.

Tarski A. [1956], *Logic, semantics and metamathematics*, Clarendon Press, Oxford.

Trillas E., Alsina C. [1993], Logic: Going farther from Tarski?, *Fuzzy Sets and Systems*, **53**, 1-13.

Trillas E., Valverde L. [1987], On inference in Fuzzy logic, *Proc. 2nd. IFSA Congress, Tokyo*, 294-297.

Turunen E. [1999], *Mathematics Behind Fuzzy Logic*, Physica-Verlag.

Valverde L. [1985], The structure of F-indistinguishability operators, *Fuzzy Sets and Systems*, **17**, 313-328.

Ward M., [1942], The closure operators of a lattice, *Annals of Mathematics*, **43**, 191-196.

Weichselberger K. , Pohlmann S. [1990], *A methodology for Uncertainty in Knowledge-Based Systems*, Springer-Verlag, Berlin.

Weston T. [1987], Approximate truth, *J. of Philosophical Logic*, **16**, 203-227.

Wójcicki R. [1970], Some remarks on the consequence operation in sentential logics, *Fund. Math*., **68**, 269-279.

Wójcicki R. [1973], Matrix approach in methodology of sentential calculi, *Studia Logica*, **32**, 7-37.

Wójcicki R. [1988], *Theory of Logical Calculi: Basic Theory of Consequence Operations*, Kluwer Academic Publishers, Dordrecht.

Ying M.S. [1994], A logic for approximate reasoning, *J. of Symbolic Logic*, **59**, 830-837

Zadeh L.A. [1965], Fuzzy Sets, *Information and Control*, **12**, 338-353.

Zadeh, L.A. [1975]a, Fuzzy logic and approximate reasoning, *Synthese*, **30**, 407-428.

Zadeh, L.A. [1975]b The concept of a linguistic variable and its application to approximate reasoning I, II, III, *Inf. Sci.* **8,** 199-275, 301-357; **9**, 43-80.

Zadeh, L.A. [1997], Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets and Systems*, **90**, 111-127.