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Giangiacomo Gerla.
Fuzzy logic: *Mathematical tools for approximate reasoning.*
Trends in Logic, Studia Logica Library 11. Kluwer Academic Publishers, 2001, xii + 269 pp.

The book under review is a very general investigation of fuzzy logic *``in narrow sense"*, mathematically deep and formal. Mostly it deals with the real unit interval as the set of truth values but with an abstract (non-structural) set of formulas, in the style of pioneering work of Pavelka in late seventies of *XX* century. Here we present some basic notions of selected chapters.

Chapter 1. - Abstract logic in a lattice. An abstract logic is a triple (L, D, M) where L is a complete lattice, $D : L \to L$ is a closure operation on L (non-decreasing, monotone w.r.t. $\leq$, idempotent) and $M \subseteq L - \{1_L\}$ called *abstract semantics*. (L, D) is called a *deduction system*. For $x \in L$, the logical consequence $L_c(x) = inf\{m \in M, x \leq m\}$.
Example: F is the set of formulas of classical logic, L is the power set of F, $D(x)$ is the set of all consequences of x .

Chapter 2. - Abstract fuzzy logic. Let F be an abstract set, call its elements formulas. An *abstract fuzzy logic* is an abstract logic whose complete lattice is the set of all fuzzy subsets of F.

Chapter 3. - Extending a crisp logic. Let (F,D) be a crisp deduction system. For v being a fuzzy subset of F let

$$D^*(v)(\alpha) = sup\{\lambda \in [0, 1] \mid \alpha \in D(C(v, \lambda))\}$$

where $C(v, \lambda)$ is the λ-cut of v, i.e., $\{x \in F, v(x) \geq \lambda\}$.

Chapter 4. - Approximate reasoning. This is ``Pavelka-style": one has a fuzzy set of formulas called *logical axioms*, a system of fuzzy deduction rules saying
``if α_1 is at least λ_1 true, . . . , α_n is at least λ_n true, then $v'(\alpha_1, \ldots, \#\alpha_n)$ is at least $v''(\alpha_1, \ldots, \alpha_n)$ true".
This gives an obvious notion of a proof from a fuzzy set v of assumptions; each proof proves its conclusion in some truth degree. The corresponding ``fuzzy Hilbert system" (F, D, M) satisfies

$$D(v)(\alpha) = \text{the supremum of degrees of proofs of } \alpha \text{ from } v.$$

The ``fuzzy semantics" M is *axiomatizable* if D equals to the logical consequence L_c given by M.

Chapter 5. - Logic as management of constraints on the truth values.

Chapter 6. - Canonical extension of a crisp Hilbert logic.

Chapter 7. - Graded consequence relations.

Chapter 8. - Truth-functional logic and fuzzy logic. The author deals with [0, 1]-valued propositional calculi, in which each connective has a truth-function; this defines the truth degree of any formula given the truth evaluation of propositional variables. He analyzes these logics using his apparatus and shows that the abstract fuzzy logic given by such a propositional logic is axiomatizable in his sense iff all truth functions are continuous. (This is a generalization of an old theorem of Pavelka.)

Chapter 9: Probabilistic fuzzy logic,

Chapter 10: Fuzzy control and approximate reasoning.

Chapter 11. Effectiveness in fuzzy logic. Here the author presents a fuzzification of main notions of recursion theory, notably: A fuzzy subset S of N (natural numbers) is recursively enumerable if there exists a recursive function h mapping $N \times N$ into the rational unit interval increasing in the second argument and such that $S(x) = lim_{n \to \infty} h(x, n)$ for each $x \in N$. Corresponding relations of arithmetical hierarchy, creative and productive sets are introduced.

These were only the selected main notions; the author proves many interesting and deep theorems in them. He also discusses relation of his notion to intuitive vague notion (e.g., the heap paradox). He takes distance from approaches to fuzzy logic in the style of (classical) truth-functional many-valued logic, e.g., he says (p. 152):
We emphasize that traditional research in truth-functional logic is sharply different from the approach to approximative reasoning we consider in this book. Indeed, in the classical approach all deductive machinery is crisp and it is devoted to produce a crisp set of formulas that are considered ``valid" for some reason.
This sounds slightly pejorative and I feel obliged to stress that the truth-functional approach to fuzzy inference, as developed e.g., in my book ***Metamathematics of fuzzy logic*** (Kluwer 1998) is (i) fully justified by the praxis of ``fuzzy logic in broad sense" and (ii) suffciently deep having absolutely natural Tarski-style semantics (giving ``some reasons" for definition of validity), leading to a very natural class of algebras of truth functions of connectives (*BL*-algebras), having ``classical" completeness for both propositional and predicate logics with this general semantics and having rather interesting properties concerning arithmetical hierarchy with respect to the ``standard" semantics given by continuous t-norms. Gerla recommends my book and similar books for reading and occasionally refers to their results. His distinction is: his logics ``arise from fuzzification of notions having metalogical characters" whereas truth-functional fuzzy logics arise from `fuzzy worlds' whose properties can be vague and therefore whose truth-value assignments can be graded. Hopefully he will agree with me that his non-truth-functional and the truth-functional approach to fuzziness are two well-founded approaches to fuzzy logic in the narrow sense and that their mutual influence is desirable.