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Grasping Infinity by Finite Sets

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Abstract. We show that the existence of an infinite set can be reduced to the existence of finite sets "as big as we will", provided that a multivalued extension of the relation of equipotence is admitted. In accordance, we modelize the notion of infinite set by a fuzzy subset representing the class of (finite) wide sets.

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1 Introduction

The main idea of this work stems from the fact that in a many-valued logic whose set of truth values is a continuum as $[0, 1]$ the meaning of an existential assertion is thoroughly different from its classical counterpart. As a matter of fact, a formula of the kind $\exists x\, A(x)$ can be valued 1 despite of the lack of an object in the interpretation domain D satisfying A. Indeed the truth value of the formula $\exists x\, A(x)$ is 1 if and only if a sequence $(d_n)_{n\in\mathbb{N}}$ of elements in D exists such that $\sup_{n\in\mathbb{N}} I(A)(d_n) = 1$, where $I(A)$ is the multivalued interpretation of the predicate A. Hence the validity of $\exists x\, A(x)$ entails the existence of objects verifying property A with a degree arbitrarily close to 1. We apply this to show that, in a multivalued environment, the existence of an infinite set can be reduced to the existence of finite sets "as big as we will", in a sense. To such an extent we consider a binary predicate EQ as a multivalued extension of the classical relation of "equipotence" between two finite sets. As in the classical case, a set is called "infinite" provided that it is equipotent to its successor, i.e. $\mathrm{INF}(x)$ denotes the formula $\mathrm{EQ}(x, x \cup \{x\})$. Equivalently, $\mathrm{INF}(x)$ can be interpreted by the fuzzy notion of "wide set". Note that the famous "sorites paradox" is based on the notion of wide set and that a very interesting solution of such a paradox using fuzzy logic was proposed by J. A. GOGUEN in [1]. Also, an interesting analysis of the connection between infinity and fuzziness was exposed by D. H. SANFORD in [6].

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2 Wide set theory

Recall some basic notions about the semantics of a multivalued logic. Let $\mathcal{L}$ be a first order language with the logical connectives $\wedge, -, \rightarrow$ and let $U = \langle [0,1], \wedge', -', \rightarrow' \rangle$ be an algebraic structure, called *evaluation structure*, whose operations are used to interpret the logical connectives. A *multivalued interpretation* or *fuzzy model* of $\mathcal{L}$ is a pair $M = \langle D, I \rangle$, where D is a set and I a map such that $I(f) : D^n \longrightarrow D$ is an n-ary operation for any name of an n-ary operation f, $I(r) : D^n \longrightarrow U$ is an n-ary fuzzy relation for any name of an n-ary predicate r, and $I(c)$ is an element of D for any constant c. Given a formula α whose free and bounded variables are within $\{x_1, x_2, \ldots, x_n\}$, we define the number $\mathrm{Val}(\alpha, d_1, \ldots, d_n)$ (where $d_1, \ldots, d_n \in D$) by induction on the complexity of α:

$$\begin{aligned}
\mathrm{Val}(r(t_1,\ldots,t_p), d_1,\ldots,d_n) &= I(r)(I(t_1)(d_1,\ldots,d_n),\ldots,I(t_p)(d_1,\ldots,d_n))\\
\mathrm{Val}(\alpha \wedge \beta, d_1,\ldots,d_n) &= \mathrm{Val}(\alpha, d_1,\ldots,d_n) \wedge' \mathrm{Val}(\beta, d_1,\ldots,d_n)\\
\mathrm{Val}(\alpha \rightarrow \beta, d_1,\ldots,d_n) &= \mathrm{Val}(\alpha, d_1,\ldots,d_n) \rightarrow' \mathrm{Val}(\beta, d_1,\ldots,d_n)\\
\mathrm{Val}(-\alpha, d_1,\ldots,d_n) &= -'\mathrm{Val}(\alpha, d_1,\ldots,d_n)\\
\mathrm{Val}(\forall x_i\, \alpha, d_1,\ldots,d_n) &= \inf_{d\in D}\{\mathrm{Val}(\alpha, d_1,\ldots,d_{i-1}, d, d_{i+1},\ldots,d_n)\}\\
\mathrm{Val}(\exists x_i\, \alpha, d_1,\ldots,d_n) &= \sup_{d\in D}\{\mathrm{Val}(\alpha, d_1,\ldots,d_{i-1}, d, d_{i+1},\ldots,d_n)\},
\end{aligned}$$

where, for any n-ary term t, the corresponding n-ary function $I(t)$ is defined as usual. We say that M *satisfies* α, and we write $M \models \alpha$, if $\mathrm{Val}(\alpha, d_1 \ldots, d_n) = 1$ for any $d_1, \ldots, d_n$ in D. Finally, given a set of formulae S, we say that M *is a model of* S and we write $M \models S$ if $M \models \alpha$ for any formula α in S. We extend I to any formula α whose free variables are among $x_1, \ldots, x_n$ by setting $I(\alpha)(d_1, \ldots, d_n) = \mathrm{Val}(\alpha, d_1, \ldots, d_n)$. So, $I(\alpha)$ is an n-ary fuzzy relation. In this paper we assume that $x \rightarrow' y$ is equal to 1 if and only if $x \leq y$, that $\wedge'$ is the standard multiplication and $-'$ is the function $1 - x$ in $[0, 1]$. The choice of the product as T-norm that interprets $\wedge$ is a kind of example. However, note that the 'minimum' is not suitable because transitive property must "weaken" the degree of equivalence. That means that, for degrees different from 1, the evaluation of conjunction $\mathrm{EQ}(x, y) \wedge \mathrm{EQ}(y, z)$ must be strictly minor than the evaluation of $\mathrm{EQ}(x, y)$ and $\mathrm{EQ}(y, z)$. We are interested to a first-order language $\mathcal{L}$ containing the usual linguistic tools for the set theory, as the relation names $\subseteq, \in, \equiv$ for the inclusion, membership relation, equipotency, the singleton-function $\{.\}$, the union $\cup$ and the constant $\emptyset$. Also, we denote by $\mathcal{L}_e$ the extension of $\mathcal{L}$ obtained by adding a binary predicate EQ we call *graded equipotency relation*. Let T be a theory in $\mathcal{L}$ expressing the main properties of the class of finite sets $\mathcal{S}$. As an example, T can be equal to $\mathsf{ZF}^* + \neg\exists x\ (x \equiv x \cup \{x\})$, where ZF^* is the theory ZF without the axiom of infinity. We add to T the following formulae in $\mathcal{L}_e$:

(1) $\mathrm{EQ}(x, x)$ (*reflexivity*),

(2) $\mathrm{EQ}(x, y) \rightarrow \mathrm{EQ}(y, x)$ (*symmetry*),

(3) $\mathrm{EQ}(x, y) \wedge \mathrm{EQ}(y, z) \rightarrow \mathrm{EQ}(x, z)$ (*transitivity*),

(4) $x \equiv x' \rightarrow (\mathrm{EQ}(x, y) \leftrightarrow \mathrm{EQ}(x', y))$ (*compatibility*).

They express the fact that EQ represents an equivalence relation extending the classical equipotence relation. If we denote by $\mathrm{INF}(x)$ the formula $\mathrm{EQ}(x, x \cup \{x\})$, it is immediate that in any classical model of set theory such a formula is satisfied by a set s if and only if s is infinite. We assume the following axioms:

(5) $\mathrm{INF}(x) \wedge (x \subseteq y) \rightarrow \mathrm{INF}(y)$ (*monotonicity*),

(6) $\neg\mathrm{INF}(\emptyset)$ ($\emptyset$ *is not infinite*),

(7) $\exists x\, \mathrm{INF}(x)$ (*infinity axiom*).

Definition 1. We denote by T^* the theory obtained by adding to T the formulae (1) through (7) and we call them *wide set theory*.

In this work we are interested only to particular models $M = \langle \mathcal{S}, I\rangle$ of T^* in which $\mathcal{S}$ is the class of finite sets of a model of ZF and $I(\epsilon)$, $I(\subseteq)$, $I(\equiv)$, $I(\emptyset)$, $I(\{.\})$ are defined as usual, i.e. they correspond to the crisp fuzzy relations in the classical model. The intended meaning of $I(\mathrm{EQ})$ is that, given two (finite) sets x and y, the number $I(\mathrm{EQ})(x, y)$ is the "degree of equipotence between x and y". Accordingly, for every $x \in \mathcal{S}$, the number $I(\mathrm{INF})(x)$ is the "degree of infinity" of x, and therefore the fuzzy set $I(\mathrm{INF})$ is the fuzzy class of "infinite sets". Equivalently, since the sets we refer to are finite, we can interpret $I(\mathrm{INF})$ as the class of the "wide sets". In this work we show that, under a suitable choice of $I(\mathrm{EQ})$, the class of finite sets is a sensible model of T^*, in particular of the infinity axiom.

3 The existence of a model

In accordance with the compatibility axiom, we assume that the degree $I(\mathrm{EQ})(x, y)$ depends only on the (classical) cardinality of x and y. In other words, we assume that there exists a suitable function $\mathrm{eq} : \mathbb{N} \times \mathbb{N} \longrightarrow [0, 1]$ in such a way that by setting

(8) $I(\mathrm{EQ})(x, y) = \mathrm{eq}(\mathrm{card}(x), \mathrm{card}(y))$

the axioms above proposed are satisfied. We call the model $\langle \mathcal{S}, I\rangle$ defined by (8) the *model associated with the function* eq. Through the following conditions the function $\mathrm{eq} : \mathbb{N} \times \mathbb{N} \longrightarrow [0, 1]$ defines a model of wide set theory.

Proposition 1. *Let* $\mathrm{eq} : \mathbb{N} \times \mathbb{N} \longrightarrow [0, 1]$ *be a fuzzy relation in* $\mathbb{N}$ *and set*

(9) $g(n) = \mathrm{eq}(n, n + 1)$.

Then the model $M = \langle \mathcal{S}, I\rangle$ *associated with* eq *is a model of wide sets theory if and only if the following properties hold for every* m, n, p *in* $\mathbb{N}$:

(i) $\mathrm{eq}(n, n) = 1$,
(ii) $\mathrm{eq}(n, m) = \mathrm{eq}(m, n)$,
(iii) $\mathrm{eq}(n, m) \geq \mathrm{eq}(n, p) \cdot \mathrm{eq}(p, m)$,
(iv) $g(n)$ *is monotonic on* $\mathbb{N}$,
(v) $g(0) = 0$,
(vi) $\lim_n g(n) = 1$.

Proof. The proof is a matter of routine: For example, to prove axiom (5), observe that $M \models \mathrm{INF}(x) \wedge (x \subseteq y) \rightarrow \mathrm{INF}(y)$ iff $\mathrm{Val}(\mathrm{INF}(x) \wedge (x \subset y) \rightarrow \mathrm{INF}(y), d_1, d_2) = 1$ for all $d_1, d_2 \in \mathcal{S}$ iff $\mathrm{Val}(\mathrm{INF}(x) \wedge (x \subseteq y), d_1, d_2) \leq \mathrm{Val}(\mathrm{INF}(y), d_2)$ for all $d_1, d_2 \in \mathcal{S}$, i.e. $\mathrm{Val}(\mathrm{INF}(x), d_1) \wedge' \mathrm{Val}(x \subseteq y, d_1, d_2) \leq \mathrm{Val}(\mathrm{INF}(y), d_2)$ for all $d_1, d_2 \in \mathcal{S}$ iff $d_1 \subseteq d_2$ implies $\mathrm{Val}(\mathrm{INF}(x), d_1) \leq \mathrm{Val}(\mathrm{INF}(y), d_2)$ iff $\mathrm{eq}(\mathrm{card}(x), \mathrm{card}(x \cup \{x\})) \leq \mathrm{eq}(\mathrm{card}(y), \mathrm{card}(y \cup \{y\}))$ for every $x \subseteq y$ iff $g(n)$ is monotonic on $\mathbb{N}$. □

We call any function eq satisfying Properties (i) through (vi) of Proposition 1 an *equipotence measure*, and we call the associated function g an *infinity measure*. In some sense, g represents the "fuzzy subset of large numbers" and, in accordance, $I(\mathrm{INF})$ is the "fuzzy subset of wide sets". Obviously, we have to prove that equipotence measures exist.

Theorem 1. *Let the function* $\bar{\mathrm{eq}}(n, m)$ *be defined by*

$$(10) \qquad \bar{\mathrm{eq}}(n, m) = \frac{\mathrm{Min}[n, m]}{\mathrm{Max}[n, m]}$$

Then $\bar{\mathrm{eq}}$ *is an equipotence measure. Therefore it exists a model of wide set theory.*

Proof. The Properties (i), (ii), (iv), (v) and (vi) are trivially satisfied. For Property (iii) consider the triple $n, m, p \in \mathbb{N}$. We can suppose that $n \leq m$. Then there are three cases:

Case 1. $p \leq n \leq m$. Then Property (iii) reduces to $\frac{n}{m} \geq \frac{p}{n}\frac{p}{m}$, which is satisfied since $n \geq p$.

Case 2. $n \leq p \leq m$. Then Property (iii) reduces to $\frac{n}{m} \geq \frac{n}{p}\frac{p}{m} = \frac{n}{m}$, which is trivially true.

Case 3. $n \leq m \leq p$. Then Property (iii) reduces to $\frac{n}{m} \geq \frac{n}{p}\frac{m}{p}$, which is true since $m \leq p$. □

We call the model defined by (10) the *base-model*, and we denote by $\bar{g}$ the function $\bar{\mathrm{eq}}(x, x+1)$, i.e. the related fuzzy set of large numbers.

4 Tuning up the infinity measure according to the context

It is a basic feature of any fuzzy concept that it is strongly dependent on the context. This is also true for the equipotence measure and the related infinity measure. For example, consider the base-model. Then, since $\bar{g}(1000) = \frac{1000}{1001}$ is very near to 1, a set with 1000 elements is "almost infinite" or, equivalently a "very large set" (and this is reasonable perhaps). None on the less, while there are contexts in which 1000 can be considered "very large", there are contexts in which we have to consider it "small". More unsatisfactory is the behavior of such an interpretation if we consider very small sets. In fact, since $\bar{g}(1) = \bar{\mathrm{eq}}(1, 2) = \frac{1}{2}$, in the base model it is not completely wrong to admit that a set with only one element is infinite. Then it is often necessary to have a large class of parametrized equipotence measures in such a way that it is possible to pick up the particular model appropriate with the order of magnitude of the context. This can be obtained starting from the base-model $\bar{\mathrm{eq}}$ and composing it with a suitable function F. So we consider *inner-expanded* functions of the kind $H_i(n, m) = \bar{\mathrm{eq}}(F(n), F(m))$ and *outer-expanded* functions of the kind $H_o = F(\bar{\mathrm{eq}}(n, m))$.

4.1 Inner-expanded functions

Let $F : \mathbb{N} \longrightarrow \mathbb{R}$ be a monotonic function such that $F(0) = 0$ and let $\bar{\text{eq}}$ be the base-model. We now analyze the conditions under which the function

$$H_i(n,m) = \bar{\text{eq}}(F(n), F(m))$$

is an equipotence measure. To do that, it is sufficient that Properties (i) through (vi) of Proposition 1 hold. Now, Properties (ii), (iii) and (vi) are trivially satisfied; Property (iv) holds by monotonicity of F. Therefore we have to impose Properties (i) and (v), i. e. that $G(n) = \dfrac{F(n)}{F(n+1)}$ is monotonic and its limit is 1.

Proposition 2. *Let $F : \mathbb{N} \longrightarrow \mathbb{R}$ be a monotonic function with $F(0) = 0$ and assume that $G(n) = \dfrac{F(n)}{F(n+1)}$ is a monotonic function such that $\lim_n G(n) = 1$. Then the function $H_i(n,m) = \bar{\text{eq}}(F(n), F(m)) = \dfrac{F(\text{Min}[n,m])}{F(\text{Max}[n,m])}$ is an equipotence measure.*

Proof. Obvious. □

The following is an immediate application of Proposition 2.

Proposition 3. *The function*

$$H_i(n,m) = \frac{\text{Min}[\log(n+1), \log(m+1)]}{\text{Max}[\log(n+1), \log(m+1)]}$$

is an equipotence function.

Proof. We refer to Proposition 2, in which $F(n) = \log(n+1)$. To prove that

$$G(n) = \frac{\log(n+1)}{\log(n+2)}$$

is an increasing function, it is sufficient to observe that the derivative is always positive. In fact

$$G'(n) = \frac{\dfrac{\log(n+1)}{n} - \dfrac{\log(n)}{n+1}}{\log^2(n+1)},$$

and, since $n\log(n)$ is increasing $(n+1)\log(n+1) - n\log(n) > 0$. The remaining Properties of Proposition 2 are immediate. □

4.2 Outer-expanded functions

Another class of equipotence functions could be obtained through the outer-expanded functions.

Proposition 4. *Let $F : [0,1] \longrightarrow [0,1]$ be a monotonic function such that $F(0) = 0$ and $F(1) = 1$ and assume that*

(a) $\lim_{x \to 1} F(x) = 1$,

(b) $F(x \cdot y) \geq F(x) \cdot F(y)$ *for any* $x, y \in [0,1]$.

Then the function

$$H_o(n,m) = F\Big(\frac{\mathrm{Min}[n,m]}{\mathrm{Max}[n,m]}\Big)$$

is an equipotence function.

Proof. Properties (i), (ii), (iv), (v) and (vi) of Proposition 1 are immediate. To prove (iii), observe that

$$\frac{\mathrm{Min}[n,m]}{\mathrm{Max}[n,m]} \geq \frac{\mathrm{Min}[n,k]}{\mathrm{Max}[n,k]} \cdot \frac{\mathrm{Min}[k,m]}{\mathrm{Max}[k,m]},$$

and, by the monotonicity of F, it follows that

$$F\Big(\frac{\mathrm{Min}[n,m]}{\mathrm{Max}[n,m]}\Big) \geq F\Big(\frac{\mathrm{Min}[n,k]}{\mathrm{Max}[n,k]} \cdot \frac{\mathrm{Min}[k,m]}{\mathrm{Max}[k,m]}\Big).$$

So, the thesis follows from (b). □

Since the notion of wideness depends on the context, it seems natural that the model M must become flexible, in some way. This can be obtained introducing a parameter λ in the model we are going to define. We call a family $(M_\lambda)_{\lambda \in \mathbb{R}^+}$ of models of T^* a *parametrized model of wide set theory* T^*.

Proposition 5. *For every* $\lambda \in \mathbb{R}^+$ *let* $M_\lambda = \langle S, I_\lambda \rangle$ *be defined by setting*

$$I_\lambda(\mathrm{EQ})(n,m) = \mathrm{eq}(n,m)^\lambda.$$

Then $(M_\lambda)_{\lambda \in \mathbb{R}^+}$ *is a parametrized model.*

Proof. We can consider I_λ as a case of outer composition $H_o(n,m)$. Since the conditions (a) and (b) of Proposition 4 are satisfied, $(M_\lambda)_{\lambda \in \mathbb{R}^+}$ is a parametrized model. □

Note that the infinity measure associated with M_λ is given by the function

$$G(n) = \Big(\frac{n}{n+1}\Big)^\lambda.$$

For example, we can find a context where a set with 10^3 elements is to be considered as "small". This can be expressed by setting, say, $G(10^5) \leq 0.4$. Solving this constraint yields $\lambda \geq 91,629$.

5 Cantor's Theorem

It is interesting to analyze whether the basic theorems of set theory are preserved in our system. As an example, we examine in this section Cantor's Theorem, claiming that the cardinality of a set is different from the cardinality of its powerset. Assume that in $\mathcal{L}$ there is also a function name P to denote the powerset $P(x)$ of a set x. Then a formula expressing Cantor's Theorem is

(11) $\quad \forall x \neg \mathrm{EQ}(x, P(x))$.

Such a formula is satisfied provided that

$$\inf_{n\in\mathbb{N}}(1-\mathrm{eq}(n,2^n)) = 1 - \sup_{n\in\mathbb{N}} \mathrm{eq}(n,2^n) = 1,$$

and therefore $\text{eq}(n, 2^n) = 0$ for every integer n. Such a condition is not satisfied in general and this is in accordance with the fact that finite sets with different cardinality may have a degree of equipotence different from 0. Therefore, Cantor's Theorem does not hold. A weaker way to express this theorem is to assert it only for "infinite" (wide) sets. Then the formula we have to consider is

(12) $\forall x\, (\text{INF}(x) \rightarrow \neg \text{EQ}(x, P(x)))$,

which is true in a multivalued model if, for every $n \in \mathbb{N}$,

(13) $\text{eq}(n, n+1) \leq 1 - \text{eq}(n, 2^n)$.

In the base-model, this is equivalent to say that

$$\frac{n}{n+1} \leq 1 - \frac{n}{2^n}.$$

Since this inequality is false for $n = 2$, in the base-model (12) it is not true. The following proposition shows that a suitable deformation of the base-model satisfies (12).

Proposition 6. *Let $F(n) = n^2$. Then the function $H_o(n,m) = F(\text{eq}(n,m))$ is an equipotence function whose associated multi-valued model satisfies* (12).

Proof. To verify (12) we have to prove that, for any integer $n \in \mathbb{N}$,

$$\frac{n^2}{(n+1)^2} \leq 1 - (\frac{n}{2^n})^2,$$

that is $2^{2n}(2n+1) \geq n^2(n+1)^2$. The latter inequality is provable by induction on n. The case $n = 1$ is immediate. As to the inductive case, we have that

$$2^{2(n+1)}(2(n+1)+1) = 4 \cdot 2^{2n}(2n+1+2) = 4 \cdot 2^{2n}(2n+1) + 8 \cdot 2^{2n}$$
$$\geq 4n^2(n+1)^2 + 2^{2n}.$$

Moreover, we have that

$$4n^2(n+1)^2 + 8 \cdot 2^{2n} \geq (n+1)^2(n+2)^2.$$

Indeed, such an inequality holds for $n = 1$. On the other hand, we have that, for any $n \geq 2$, $4n^2 \geq (n+2)^2$. Thus, the chain of inequalities holds and so

$$2^{2(n+1)}(2(n+1)+1) \geq (n+1)^2(n+2)^2,$$

and the step of induction is proved. Hence the thesis follows. □

Finally, we can simply assert the existence of an infinite set satisfying Cantor's Theorem. The corresponding formula is

(14) $\exists x\, (\text{INF}(x) \wedge \neg \text{EQ}(x, P(x)))$.

The following proposition shows that (14) does not depend on the system T^* of axioms.

Proposition 7. *The base model satisfies* (14). *Also, there is a model of wide set theory that does not satisfy* (14).

Proof. Let $F(n) = \dfrac{n}{n+1}$ and consider the corresponding inner-expanded multivalued model M. Since the Properties of Proposition 2 are verified for F, M is a model of wide set theory. Formula (14) is true if the following condition is satisfied:

(15) $\sup_{n \in \mathbb{N}}(\text{eq}(n, n+1)(1 - eq(n, 2^n))) = 1$.

If $H_i(n,m) = \bar{\mathrm{eq}}(F(n), F(m))$ is an inner-expanded model, (15) becomes

(16) $\sup_{n\in\mathbb{N}}\left(\frac{F(n)}{F(n+1)}\left(1-\frac{F(n)}{F(2^n)}\right)\right) = 1.$

Since

$$\lim_n \frac{F(n)}{F(n+1)}\left(1-\frac{F(n)}{F(2^n)}\right) = 0,$$

the set $\{n \in \mathbb{N} : \frac{F(n)}{F(n+1)}\left(1-\frac{F(n)}{F(2^n)}\right) > \frac{1}{2}\}$ is finite. Let K be the maximum of such set. Moreover, for any integer $n \in \mathbb{N}$,

$$\frac{F(n)}{F(n+1)}\left(1-\frac{F(n)}{F(2^n)}\right) < 1.$$

Therefore,

$$\sup_n\left(\frac{F(n)}{F(n+1)}\left(1-\frac{F(n)}{F(2^n)}\right)\right) = K < 1.$$

Thus, (14) is not true in the model M. □

6 Degree of inclusion and degree of equipotency

In this section we investigate on the connections between a multivalued version of inclusion and the notion of equipotence measure. We start from KOSKO's definition of conditional probability in terms of a fuzzy inclusion (see [4]) and we show that it leads to the canonical multivalued model of the equipotence relation. Conversely, an equipotence relation defines straightforwardly a fuzzy inclusion.

We add to the language $\mathcal{L}$ a binary predicate INC for a multivalued inclusion relation. Following the approach of KOSKO, an interpretation for INC can be obtained through conditional probability by setting $I(\mathrm{INC})(\emptyset, y) = 1$ and

(17) $I(\mathrm{INC})(x,y) = \frac{\mathrm{card}(x\cap y)}{\mathrm{card}(x)}$, for every $x \neq \emptyset$.

Now, in classical set theory, a set x has lesser cardinality than a set y if there is a "copy" x' of x such that $x' \subseteq y$. This leads to the predicate $\exists x' (x' \equiv x \wedge \mathrm{INC}(x', y))$ that we denote by $\mathrm{PM}(x,y)$. Such a predicate, when evaluated on a multivalued model M, gives the following fuzzy relation

(18) $I(\mathrm{PM})(x,y) = \sup_{x'\in S}\{I(\mathrm{INC})(x',y) : x' \equiv x\}.$

The following proposition is immediate.

Proposition 8. *Let I be Kosko's interpretation of* INC. *Then, for any x and y,*

$$I(\mathrm{PM})(x,y) = \begin{cases} 1 & \textit{if } \mathrm{card}(x) \leq \mathrm{card}(y), \\ \frac{\mathrm{card}(y)}{\mathrm{card}(x)} & \textit{if } \mathrm{card}(x) \geq \mathrm{card}(y). \end{cases}$$

We are now ready to define the predicate EQ_1: We write $\mathrm{EQ}_1(x,y)$ to denote $\mathrm{PM}(x,y) \wedge \mathrm{PM}(x,y)$. Obviously, in a multivalued model $M = \langle D, I\rangle$,

(19) $I(\mathrm{EQ}_1)(x,y) = I(\mathrm{PM})(x,y)\cdot I(\mathrm{PM})(y,x).$

The following theorem is immediate.

Theorem 2. *The interpretation of* EQ_1 *in Kosko's model coincides with the interpretation of* EQ *in the base-model.*

On the other hand, once we have the predicate EQ as primitive, since in the classical case $x \subseteq y$ iff $x \cap y = x$, we can define the predicate $\mathrm{INC}(x, y)$ by the formula $\mathrm{EQ}(x \cap y, x)$. The proof of the following theorem is immediate.

Theorem 3. *Consider the base model and let* $\mathrm{INC}(x, y)$ *denote the formula* $\mathrm{EQ}(x \cap y, x)$. *Then* $I(\mathrm{INC})$ *is Kosko's inclusion.*

Note that KOSKO's definition of inclusion suffers a major drawback: seen as a fuzzy relation, it is not transitive. Indeed, if we take three sets $x, y, z \in \mathcal{S}$ such that $x \cap z = \emptyset$, $x \cap y \neq \emptyset$, and $y \cap z \neq \emptyset$, the predicate $\mathrm{INC}(x, y) \wedge \mathrm{INC}(y, z) \rightarrow \mathrm{INC}(x, z)$ is not true in the model of conditional probabilities.

7 Fuzzy equivalence relations and pseudo-metric spaces

In this section we consider a "duality" between equipotence measures and a particular class of extended pseudo-metric spaces. The relations between pseudo-metrics and fuzzy equivalence relations are straightforward and some examples can be found in [2] and [3]. Recall that an *extended pseudo-metrics on a set* S is an application $d : S \times S \longrightarrow [0, +\infty]$ such that

(i) $d(x, x) = 0$ for all $x \in S$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in S$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in S$.

We say that an extended pseudometric d on the set of integers $\mathbb{N}$ is *asymptotically compressed* if $f(n) = d(n, n+1)$ is a decreasing function such that $\lim_n d(n, n+1) = 0$. Also, recall that a fuzzy relation $\mathcal{R}$ over a set S is called a *fuzzy equivalence* provided that the following properties hold:

(i) $\mathcal{R}(x, x) = 1$,
(ii) $\mathcal{R}(x, y) = \mathcal{R}(y, x)$ for any $x, y \in S$,
(iii) $\mathcal{R}(x, y) \geq \mathcal{R}(x, z) \cdot \mathcal{R}(z, y)$ for any $x, y, z \in S$.

We extend the functions $\log x$ and e^x by setting $\log 0 = -\infty$, $-\log 0 = \infty$ and $\mathrm{e}^{-\infty} = 0$.

Proposition 9. *Let* $\mathcal{R}$ *be a fuzzy equivalence on the set* S *and define, for any* $x, y \in S$,

(20) $$d_{\mathcal{R}}(x, y) = -\log(\mathcal{R}(x, y)).$$

Then $d_{\mathcal{R}}$ *is an extended pseudometric on* S.

Conversely, let d *be an extended pseudometric over* S. *Then the following fuzzy relation* $\mathcal{R}_d$ *over* S

(21) $$\mathcal{R}_d(x, y) = \mathrm{e}^{-d(x,y)}$$

is a fuzzy equivalence.

Proof. Since $\mathcal{R}(x,y) \in [0,1]$, the range of $d_{\mathcal{R}}$ is $[0,+\infty]$. Moreover, since $\mathcal{R}(x,x) = 1$, $d_{\mathcal{R}}(x,x) = 0$. Besides, $d_{\mathcal{R}}(x,y) = d_{\mathcal{R}}(y,x)$ trivially. Finally, for any $x,y,z \in S$,

$$\begin{aligned} d_{\mathcal{R}}(x,y) \le d_{\mathcal{R}}(x,y) + d_{\mathcal{R}}(y,z) \ &\text{iff} \ -\log(\mathcal{R}(x,y)) \le -(\log(\mathcal{R}(x,y)) + \log(\mathcal{R}(y,z)) \\ &\text{iff} \ -\log(\mathcal{R}(x,y)) \le -(\log(\mathcal{R}(x,y) \cdot \mathcal{R}(y,z)) \\ &\text{iff} \ \mathcal{R}(x,y) \ge \mathcal{R}(x,y) \cdot \mathcal{R}(y,z). \end{aligned}$$

Since this is true by (iii), the function $d_{\mathcal{R}}$ is an extended pseudometric over $S \times S$. Conversely, let d be an extended pseudometric over S. Since $d(x,x) = 0$, we have that $\mathcal{R}_d(x,x) = 1$. Besides, $\mathcal{R}_d(x,y) = \mathcal{R}_d(y,x)$, since $d(x,y) = d(y,x)$. Finally, $\mathcal{R}_d(x,y) \ge \mathcal{R}_d(x,z) \cdot \mathcal{R}_d(z,y)$, since $d(x,y) \le d(x,z) + d(z,y)$. □

Note that given a fuzzy equivalence $\mathcal{R}$ over S it may well be the case that $\mathcal{R}(x,y) = 0$ for some $x,y \in S$. In this case, $d_{\mathcal{R}}(x,y) = \infty$, and this explains why the range of $d_{\mathcal{R}}$ is assumed to be $[0,+\infty]$. Equations (20) and (21) are remarkable to the extent that they establish a duality between fuzzy equivalence relations and extended pseudometrics. The following proposition extends this duality to equipotence measures and compressed extended pseudometrics.

Theorem 4. *Consider an equipotence measure* eq *and let* d_{eq} *be defined by* (20). *Then* d_{eq} *is an asymptotically compressed extended pseudometric in* $\mathbb{N}$ *such that* $d_{\text{eq}}(0,1) = +\infty$.

Proof. Since $d_{\text{eq}}(n,m) = -\log(\text{eq}(n,m))$ and $\text{eq}(n,n+1)$ is an increasing function, it follows that $d_{\text{eq}}(n,n+1)$ is decreasing. Moreover, since $\lim_n \text{eq}(n,n+1) = 1$, $\lim_n d_{\text{eq}}(n,n+1) = 0$. Finally, $d_{\text{eq}}(0,1) = -\log(\text{eq}(0,1))) = -\log(0) = +\infty$. □

Theorem 5. *Let d be an asymptotically compressed extended pseudometric over* $\mathbb{N}$ *such that* $d(0,1) = +\infty$. *Then the relation* $\mathcal{R}_d(n,m) = \mathrm{e}^{-d(n,m)}$ *is an equipotence function.*

Proof. We prove that Properties (i) through (vi) of Proposition 1 are satisfied. Properties (i) through (iii) are true, since $\mathcal{R}_d$ is a pseudometric on $\mathbb{N}$ as follows

$$d(x,y) = \begin{cases} |\sqrt{x} - \sqrt{y}| & \text{if } x \ne 0,\ y \ne 0, \\ 0 & \text{if } x = y = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Then, the associated fuzzy relation $R_d(x,y) = \mathrm{e}^{-d(x,y)}$ is an equipotence measure, as it is easy to see. □

8 Conclusions and future developments

The topics discussed so far seem to have a larger scope than the one of this work. As a matter of fact, it seems that every process of approximation through which we can obtain objects verifying a certain property A with a degree close to truth "as much as we" could be turned into the validity of the formula $\exists x\, A(x)$ in a multivalued logic. Nevertheless, the kind of existence supported by multivalued logic could appear too unsatisfactory to those willing the actual exhibition of an object that fully satisfies the required property. Therefore, it would be appropriate to single out a multivalued logic such that a theorem of the following kind holds: *Given a multivalued model M it* *is possible to build up a classical model M' such that any existential formula which is true in M (i. e. possesses the designated value 1) is true in M', too.* Provided this, the validity in M of a formula $\exists x\, A(x)$ (derived from the existence of an approximation process) entails the validity of such a formula in the classic model M' and therefore the existence of an object verifying $A(x)$. In other words, a process entailing the potential construction of an object could be turned into the actual exhibition of the object. But this is a subject of investigation that will be tackled in an upcoming work.

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