Connecting Fuzzy submonoids, fuzzy preorders and quasi-metrics.

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Abstract

This paper is an extended abstract of my paper [12] published in *Fuzzy Set and Systems*. We start from a residuated lattice L and a monoid M, and we define a Galois connection from the lattice of the compatible L-preorders in M and the lattice of L-submonoids of M. Given a set S we define a Galois connection between the lattice of the L-preorders in S and the lattice of L-submonoids of the monoid $(S^S, \circ, i)$. A link with the notion of quasi-metric is also established.

Keywords. Fuzzy monoids, fuzzy orders, similarities, quasi-metric spaces, residuated lattices.

1 Introduction

Rosenfeld, in its pioneer work [21] gives the very interesting notion of *fuzzy subgroup* of a given group. Such a notion was extended into the general notion of *fuzzy subalgebra* of an algebraic structure (see, for example, [2], [6] and [7]). Another basic notion is the one of *fuzzy equivalence* extending the classical one of equivalence (see for example, Chakraborty and Das [3], Valverde and Jacas [14], [23] and Ovchinikov [20]). In [19] the notion of a fuzzy subgroup of a group G is related with the one of fuzzy equivalence in

G. In [8] and [13] one shows that, given a nonempty set S, there is a Galois connection among the lattice of the fuzzy equivalences in S, and the lattice of the fuzzy subgroups of transformations in S.

This paper is devoted to propose an analogous connection between the lattice of the compatible L-preorders and the lattice of the L-submonoids of a monoid. In particular, we refer to the monoid $(S^S, \circ, i)$ of the maps from S into S. In account of a well known duality between fuzzy orders and quasi-metrics, this induces also a Galois connection between the lattice of the quasi-metric on S and the lattice of the fuzzy submonoids of S^S. We focalize our attention also to formulas to calculate the L-submonoid generated by a given L-subset and the L-order generated by an L-relation. Some examples are given and some suggestions for future applications are also outlined in the field of fuzzy codes theory, complexity theory and genetics.

2 Preliminaries

In this paper L always denotes a complete residuated lattice $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ (see [5]). This means that:

- $(L, \wedge, \vee, 0, 1)$ is a complete lattice
- $*$ is a commutative and associative operation such that $x * 1 = x$
- $z \leq x \rightarrow y \Leftrightarrow x * z \leq y$ (adjointness).

It is easy to prove that in any complete residuated lattice the infinite distributivity

$$(\sup_{i\in I} x_i) * x = \sup_{i\in I}(x_i * x)$$

holds true. More precisely, in a complete lattice such a property is equivalent to the existence of residuum for $*$. The intended interpretation is that L is the set of truth values of a multi-valued logic and that $*$ and $\rightarrow$ are the interpretations of the logical connectives "and" and "implies", respectively. Given a nonempy set S, we denote by L^S the direct power of L and we call *L-subset* of S any element in L^S, i.e. any map from S to L. An *L-relation* is an L-subset of a cartesian product. The direct power L^S is a complete lattice. We call *inclusion relation* the order relation in L^S and we denote it by $\subseteq$. We call *intersection* and *union* the meet and join operations in L^S and we denote them by $\cap$ and $\cup$, respectively. We say that an L-subset s is *crisp* provided that $s(x) \in \{0,1\}$ for every $x \in S$. By associating any subset of S with the related characteristic function, we can identify the subsets of S with the crisp L-subsets of S. As an example, we identify the empty set with the map s_0 constantly equal to 0 and S with the map s_1 constantly equal to 1. This gives an embedding of the lattice $(P(S), \cap, \cup, \emptyset, S)$ into the lattice $(L^S, \cap, \cup, s_0, s_1)$.

We conclude this section by recalling some basic definitions in ordered set theory.

Definition 1. Let L be a complete lattice, then a *closure system* in L is any class $\mathcal{C}$ of elements of L such that the meet of any family of elements of $\mathcal{C}$ is an element of $\mathcal{C}$. Given $x \in L$, we say that

$$\bar{x} = \inf\{z \in \mathcal{C} \mid z \geq x\}$$

is the element in $\mathcal{C}$ *generated by* x.

Any closure system is a complete lattice in which the join of a family $(x_i)_{i \in I}$ of elements is the element generated by $\sup_{i \in I} x_i$ and the meet is $\inf_{i \in I} x_i$.

Definition 2. Let L be a complete lattice, then an order-preserving map $H : L \to L$ is called a *closure operator* provided that

$$H(x) \geq x \quad ; \quad H(H(x)) = H(x).$$

It is easy to prove that if H is a closure operator, then the set $\{x \in L : H(x) = x\}$ of fixed points of H is a closure system. Conversely, if $\mathcal{C}$ is a closure system, then by setting $H(x) = \overline{x}$ we obtain a closure operator. The order-theoretically dual notions are *interior system*, *interior operator* and *interior* of an element x. As an example, a map H is an interior operator provided that it is order preserving and

$$H(x) \leq x \quad ; \quad H(H(x)) = H(x).$$

Definition 3. Let L_1 and L_2 be complete lattices, then a *Galois connection* from L_1 to L_2 is a pair (h, k) of order-preserving maps $h : L_1 \to L_2$ and $k : L_2 \to L_1$ such that
- $k \circ h : L_1 \to L_1$ is an interior operator in L_1,
- $h \circ k : L_2 \to L_2$ is a closure operator in L_2.
In the case $k \circ h$ and $h \circ k$ are identity maps, we say that (h, k) is a *lattice isomorphism*.

We represent a Galois connection as follows:

$$L_1 \underset{k}{\overset{h}{\rightleftarrows}} L_2$$

Equivalently, (h, k) is a Galois connection if and only if h and k are order-preserving maps such that for all $x \in L_1$ and for all $y \in L_2$

$$k(h(x)) \leq x \quad ; \quad h(k(y)) \geq y.$$

Observe that there are several equivalent definitions of Galois connection we obtain by referring to the duals of the considered lattices. As an example, if we consider the dual of L_1, then we can define a Galois connection as a pair (h, k) such that h and k are order-reversing and both $h \circ k$ and $k \circ h$ are closure operators. It is easy to prove that if (h, k) is a Galois connection from L_1 to L_2 and (h', k') is a Galois connection from L_2 to L_3, then the *composition* $(h' \circ h, k \circ k')$ is a Galois connection from L_1 to L_3.

3 Fuzzy submonoids and L-preorders

The first notion we consider in this paper is the one of L-submonoids.

Definition 4. Let $M = (M, \cdot, e)$ be a monoid. Then an *L-submonoid* of M is an L-subset m of M such that

i) $m(e) = 1$,

ii) $m(x \cdot y) \geq m(x) * m(y)$ for every $x, y \in M$.

If we consider a multivalued logic whose language contains a monadic predicate symbol $\widetilde{M}$, the constant e and the operation symbol $\cdot$, then m is an L-submonoid of M if and only if $(M, \cdot, m, e)$ is a multi-valued model of the axioms

$$\widetilde{M}(e) \;\; ; \;\; \forall x \forall y (\widetilde{M}(x) \wedge \widetilde{M}(y) \to \widetilde{M}(x \cdot y)).$$

So, the just given definition is the natural extension in a multi-valued logic of the definition of monoid with a given submonoid. The following proposition gives a formula to obtain the L-submonoid generated by a given L-subset of M.

Proposition 5. *The class $L-SM(M)$ of L-submonoids of a monoid $(M, \cdot, e)$ is a closure system in the lattice of L-subsets of M. Given an L-subset s of M, the L-submonoid $\overline{s}$ generated by s can be obtained by setting $\overline{s}(e) = 1$ and,*

$$\overline{s}(x) = sup\{s(x_1) * ... * s(x_n) : x_1 \cdot ... \cdot x_n = x\}$$

in the case $x \neq e$.

Proof. See [12]. ∎

The second notion we are interested is the one of L-preorder.

Definition 6. An *L-preorder* on a nonempty set S is an L-relation $r : S \times S \to L$ such that:

i) $r(x, x) = 1$ for every $x \in S$ (reflexivity),

ii) $r(x, z) \geq r(x, y) * r(y, z)$ for every $x, y, z \in S$ ($*$-transitivity).

If we consider a multi-valued logic with a binary predicate symbol $\widetilde{R}$, then r is an L-preorder in S if and only if (S, r) is a multi-valued model of the axioms

$$\forall x \widetilde{R}(x, x) \;\; ; \;\; \forall x \forall y \forall z (\widetilde{R}(x, y) \wedge \widetilde{R}(y, z) \to \widetilde{R}(x, z)).$$

So, the just given definition is the natural extension in multi-valued logic of the definition of preorder. The following proposition gives a formula for the L-preorder generated by a given L-relation in S (see [1]).

Proposition 7. *The class $L - PO(S)$ of the L-preorders on S is a closure system. For any L-relation $r : S \times S \to L$, the L-relation $\overline{r}$ obtained by setting $\overline{r}(x, x) = 1$ and, in the case $x \neq y$,*

$$\overline{r}(x, y) = \sup\{r(x_1, x_2) * ... * r(x_{n-1}, x_n) : x_1 = x, x_n = y\},$$

is the L-preorder generated by r.

Proof. See [12]. ∎

An obvious example of L-preorder is the residuation $\to$. More generally, in [23] one proves the following proposition.

Proposition 8. *Any L-subset s of S defines an L-preorder r_s obtained by setting*

$$r_s(x, y) = s(x) \to s(y)$$

for any $x, y \in S$. The class of so defined L-preorders is a basis in $L-PO(S)$, i.e. any L-preorder is intersection of so defined L-preorders.

4 A link between L-preorders and L-submonoids

In this section we will examine some links between L-preorders and L-submonoids. Let $(M, \cdot, e)$ be a monoid, then we say that an L-relation r in M is *right compatible*, in brief *compatible*, provided that

$$r(x, y) \leq r(x \cdot t, y \cdot t)$$

for any $t \in M$ (see [19]). The residuation is an example of compatible L-relation.

Proposition 9. *The class of compatible L-relations in a monoid M is a closure system and therefore a complete lattice. Let r be an L-relation in M, then by setting*

$$\widetilde{r}(x, y) = \sup\{r(x', y') : \exists t \in M, x' \cdot t = x,\ y' \cdot t = y\},$$

we obtain the compatible L-relation generated by r. The class $L - CPO(M)$ of compatible L-preorders in M is a closure system. Moreover, given an L-relation r, $\overline{\widetilde{r}}$ is the compatible L-preorder generated by r.

Proof. See [12]. ∎

Definition 10. Let $(M, \cdot, e)$ be a monoid. Then we associate any L-relation r in M with the L-subset $\alpha_1(r)$ defined by setting

$$\alpha_1(r)(x) = r(e, x) \tag{1}$$

for any $x \in M$. Also, we associate any L-subset m of M, with the L-relation $\beta_1(m)$ defined by setting

$$\beta_1(m)(x, y) = \sup\{m(t) : t \cdot x = y\}. \tag{2}$$

The interest of these definitions lies in the following proposition.

Proposition 11. *Let $(M, \cdot, e)$ be a monoid and r a compatible L-preorder. Then $\alpha_1(r)$ is an L-submonoid. Let m be an L-submonoid of M, then $\beta_1(m)$ is a compatible L-preorder.*

Proof. See [12]. ∎

Now, we are ready to prove the following theorem.

Theorem 12. *Both α_1 and β_1 are order-preserving maps with respect to the inclusion relation. Moreover, for any $r \in L-CPO(M)$ and $m \in L-SM(M)$*

$$\beta_1(\alpha_1(r)) \subseteq r \quad ; \quad \alpha_1(\beta_1(m)) = m.$$

Consequently, (α_1, β_1) is a Galois connection from $L - CPO(M)$ into $L - SM(M)$

$$L - CPO(M) \quad \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \quad L - SM(M) \ .$$

Proof. See [12]. ■

Observe that $\beta_1(\alpha_1(r)) \neq r$, in general. As an example, set $r = M \times M$. Then $\alpha_1(M \times M) = M$ and therefore $\beta_1(\alpha_1(M \times M)) = \{(x, y) : \exists t \in M, x \cdot t = y\}$. Assume that $a, b \in M$ exists such that a is not a left divisor of b, then (a, b) is not an element in $\beta_1(\alpha_1(M \times M))$ and therefore $\beta_1(\alpha_1(M \times M)) \neq M \times M$.

Proposition 13. *Let s be an L-subset of M, and $\overline{s}$ is the L-submonoid generated by s. Then*

$$\beta_1(\overline{s})(x, y) = \sup\{s(t_1) * ... * s(t_n) : t_1 \cdot ... \cdot t_n \cdot x = y\}. \tag{3}$$

Let r be an L-relation in M and let $\overline{\widetilde{r}}$ be the compatible L-preorder generated by r. Then

$$\alpha_1(\overline{\widetilde{r}})(x) = \sup\{\widetilde{r}(e, x_1) * ... * \widetilde{r}(x_n, x) : x_1, ..., x_n \in S\}. \tag{4}$$

In other words (3) enables us to define an L-preorder $\beta_1(\overline{s})$ in M from any L-subset s of M and (4) enables us to define an L-submonoid $\alpha_1(\overline{\widetilde{r}})$ of M from any L-relation in M.

5 L-submonoids of $(S^S, \circ, i)$

If S is a nonempty set, then we can consider the monoid $(S^S, \circ, i)$ where S^S is the class of all the maps from S into S, $\circ$ is the composition operator and i is the identity map. It is not restrictive to concentrate our attention on such a kind of monoid since any monoid $(M, \cdot, e)$ is isomorphic with a suitable submonoid of $(M^M, \circ, i)$. Indeed, the map $h : M \to M^M$ defined by setting, for any $a \in M$, $h(a)(x) = a \cdot x$ is an injective homomorphism. A trivial submonoid of S^S is defined by the set $Const = \{c_x \in S^S : x \in S\} \cup \{i\}$ where, for any $x \in S$, we denote by c_x the map constantly equal to x. The proof of the following proposition is well known.

Proposition 14. *Let r be an L-preorder on S and define the L-relation $\sigma(r)$ in S^S by setting*

$$\sigma(r)(f, g) = \inf_{x \in S} r(f(x), g(x)) \tag{5}$$

for all $f, g \in S^S$. Then $\sigma(r)$ is a compatible L-preorder. Let r' be a compatible L-preorder in S^S and define the L-relation $\tau(r')$ in S by setting

$$\tau(r')(x, y) = r'(c_x, c_y) \tag{6}$$

for every $x, y \in S$. Then $\tau(r')$ is an L-preorder in S.

Notice that, in a sense, $\sigma(r)(f, g)$ is a valuation in a multi-valued logic of the claim that for any $x \in S$ the images $f(x)$ and $g(x)$ are in the relation r.

Proposition 15. *The maps $\sigma : L - PO(S) \to L - CPO(S^S)$ and $\tau : L - CPO(S^S) \to L - PO(S)$ are order-preserving. Moreover*

$$\tau(\sigma(r)) = r \quad ; \quad \sigma(\tau(r')) \supseteq r'$$

for any $r \in L - PO(S)$ and $r' \in L - CPO(S^S)$. As a consequence, the pair (σ, τ) is a Galois connection from $L - PO(S)$ to $L - CPO(S^S)$

$$L - PO(S) \quad \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \quad L - CPO(S^S) \ .$$

Proof. See [12]. ∎

Observe that $\sigma(\tau(r')) \neq r'$, in general. Indeed, assume that S is equal to the real number set and define $r' : S^S \to \{0, 1\}$ by setting $r'(f, g) = 1$ provided that $f = g$ or $f(S) \leq g(S)$, i.e. $f(x) \leq g(y)$ for any $x, y \in S$. Then r' is a compatible L-preorder, $\tau(r')$ coincides with the natural order in S and $\sigma(\tau(r'))$ with the pointwise order in the functional space S^S. As a consequence, $\sigma(\tau(r')) \neq r'$.

Definition 16. We denote by (α_1', β_1') the Galois connection from $L-PO(S)$ to $L-SM(S^S)$ obtained by composing (σ, τ) with (α_1, β_1)

$$L-PO(S) \underset{\beta_1' = \tau \circ \beta_1}{\overset{\alpha_1' = \alpha_1 \circ \sigma}{\rightleftarrows}} L-SM(S^S) \ .$$

Observe that

$$\alpha_1'(r)(f) = \inf_{x \in S} r(x, f(x)) \tag{7}$$

and

$$\beta_1'(m)(x,y) = \sup \left\{ m(f) \mid f(x) = y, f \in S^S \right\}. \tag{8}$$

So, we can consider $\alpha_1'(r)(f)$ as the valuation in a multivalued logic of the claim that $f(x)$ is greater than or equal to x for all the elements $x \in S$ and $\beta_1'(m)(x,y)$ as the valuation of the claim that a function f in m exists such that $f(x) = y$.

Proposition 17. *For every L-preorder r in S and L-submonoid m of S^S,*

$$\beta_1'(\alpha_1'(r)) = r \ ; \ \alpha_1'(\beta_1'(m)) \supseteq m. \tag{9}$$

Proof. See [12]. ∎

It is not possible to prove that $\alpha_1'(\beta_1'(m)) = m$. A counterexample is obtained by noticing that $\beta_1'(Const) = S \times S$ and therefore $\alpha_1'(\beta_1'(Const)) = S^S$.

Example. Let s be an L-subset of S to represent a vague property. Then, by considering the associated L-order r_s we obtain

$$\alpha_1'(r_s)(f) = \inf_{x \in S}(s(x) \rightarrow s(f(x))).$$

In other words, $\alpha_1'(r_s)$ is the L-monoid of the transformations preserving the vague propery s. In accordance with the fact that the intersection of a family of L-submonoids is an L-submonoid, it is possible to extend such an example to any class of vague properties.

Example. Consider an L-subset *small*: $S^S \to L$ of S^S we interpret as the L-subset of "small" ("possible", "non expensive", "legal", ...) transformations. Then, in accordance with (3), an L-ordering r is defined by setting

$$r(x, y) = \sup\{small(f_1) * ... * small(f_n) : f_1(...f_n(x)...) = y\}. \tag{10}$$

We can imagine that $r(x, y)$ is the valuation in a multi-valued logic of the claim that it is possible to transform x into y by a suitable sequence of "small" (equivalently "possible", "non expensive", "legal",...) transformations.

6 Quasi-metrics and metrics.

In this section we refer to a residuated lattice $([0, 1], \wedge, \vee, *, \rightarrow, 0, 1)$ where $*$ is a continuous *Archimedean* t-norm, i.e. a continuous t-norm such that $x * x < x$ for any $x \in (0, 1)$. These norms satisfy a basic representation theorem (see for example [16]). Indeed, denote by $([0, +\infty], \leq, +, 0)$ the extension of the ordered monoid $([0, +\infty), \leq, +, 0)$ defined by assuming that, for any $x \in [0, +\infty]$, $x + (+\infty) = (+\infty) + x = +\infty$, $x \leq \infty$. Then we call *additive generator*, any continuous strictly decreasing function $h : [0, 1] \to [0, +\infty]$ such that $h(1) = 0$ and we denote by $h^{[-1]}$ the map defined by

$$h^{[-1]}(x) = \begin{cases} h^{-1}(x) & \text{if } x \in h([0, 1]) \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Observe that, since $h([0, 1])$ is the interval $[0, h(0)]$, $h^{[-1]}(x) = h^{-1}(x)$ if $x \leq h(0)$ and 0 otherwise. In particular, for any $x \in [0, 1]$ and $y \in [0, +\infty]$

$$h^{[-1]}(h(x)) = x \quad ; \quad h(h^{[-1]}(y)) = y \wedge h(0).$$

Proposition 18. *A binary operation $*$ is a continuous Archimedean t-norm if and only if an additive generator h exists such that for all $x, y \in [0, 1]$*

$$x * y = h^{[-1]}(h(x) + h(y)). \tag{12}$$

Also, $*$ is strict, i.e. strictly increasing with respect to both the variables, if and only if $h(0) = \infty$.

For example, if we set $h(x) = -\ln(x)$ for any $x > 0$ and $h(0) = \infty$, then we obtain a generator whose associated t-norm is the usual product. If $h(x) = 1 - x$, then $*$ is the t-norm of Lukasiewicz. The notion of additive generator is on the basis of a duality established by Valverde in [23] and enables us to restate all the results in this paper in metrical terms.

Definition 19. Given a nonempty set S, we call (*extended*) *quasi-pseudo-metric* on S any map $d : S \times S \to [0, +\infty]$ such that, for any $x, y, z \in S$,

$$d(x, x) = 0, \qquad d(x, y) \leq d(x, z) + d(z, y).$$

The class of these maps is denoted by $QPM(S)$.

Such a notion extends the one of metric space since a metric space is an extended quasi-pseudo-metric space which is symmetric, with real values and separating, i.e. $d(x, y) = 0$ entails $x = y$. In the following we consider the lattice $([0, +\infty]^{S\times S}, \preceq)$ where $\preceq$ is defined by setting $d_1 \preceq d_2$ provided that $d_2(x, y) \leq d_1(x, y)$ for any $x, y \in S$.

Proposition 20. *The class $QPM(S)$ of all extended quasi-pseudo-metric on S is a closure system in $([0, +\infty]^{S\times S}, \preceq)$ and therefore a complete lattice. Also, for any $d \in [0, +\infty]^{S\times S}$, the quasi-pseudo-metric generated by d is the function $\overline{d}$ defined by setting $\overline{d}(x, x) = 0$ and, in the case $x \neq y$,*

$$\overline{d}(x, y) = \inf\{d(x_1, x_2) + ... + d(x_{n-1}, x_n) : x_1, ..., x_n \in S, x_1 = x, x_n = y\}.$$

Proof. See [12]. ■

In this section we assume that h is a generator of a continuous Archimedean t-norm $*$ and that L the related residuated lattice. The following proposition was proved in [23].

Proposition 21. *We can associate any extended quasi-pseudo-metric d with the L-preorder $\alpha_0(d) : S \times S \to [0, 1]$ defined by setting*

$$\alpha_0(d)(x, y) = h^{[-1]}(d(x, y)). \tag{13}$$

Conversely, we can associate any L-preorder r with an extended quasi-pseudo-metric $\beta_0(r) : S \times S \to [0, \infty]$ defined by setting

$$\beta_0(r)(x, y) = h(r(x, y)). \tag{14}$$

The proof of the following theorem is immediate.

Proposition 22. *Both the maps $\alpha_0 : QPM(S) \to L - PO(S)$ and $\beta_0 : L - PO(S) \to QPM(S)$ are order-preserving and*

$$\alpha_0(\beta_0(r)) = r \quad ; \quad \beta_0(\alpha_0(d)) = d \wedge h(0) \leq d. \tag{15}$$

Observe that we are not authorized to claim that the pair (α_0, β_0) is a Galois connection from $QPM(S)$ to $L - PO(S)$ since in the space $QPM(S)$ we consider the dual order. To obtain this, we have to confine ourselves to the strict Archimedean t-norms.

Theorem 23. *Assume that $h(0) = +\infty$, then*

$$\alpha_0(\beta_0(r)) = r \quad ; \quad \beta_0(\alpha_0(d)) = d. \tag{16}$$

As a consequence (α_0, β_0) is an isomorphism and therefore a Galois connection from $QPM(S)$ to $L - PO(S)$,

$$QPM(S) \quad \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \quad L - PO(S)\ .$$

In this section we consider only Archimedean strict t-norms. As an alternative, it should be considered any generator h and the restriction of α_0 to the sublattice of the elements in $QPM(S)$ bounded by $h(0)$.
We say that an extended quasi-metric d in a monoid M is *right compatible*, in brief *compatible* provided that

$$d(x, y) \geq d(x \cdot t, y \cdot t)$$

and we denote by $CQPM(M)$ the class of compatible extended quasi-pseudo-metrics in M. The proof of the following proposition is trivial.

Theorem 24. *Let M be a monoid. Then the pair (α_0, β_0) is a Galois connection from $CQPM(M)$ to $L - CPO(M)$*

$$CQPM(M) \quad \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \quad L - CPO(M)\ .$$

In accordance with the fact that the composition of two Galois connections is a Galois connection, we can give the following definitions.

Definition 25. We denote by (α_2, β_2) the Galois connection from $CQPM(M)$ to $L - SB(M)$ obtained by composing the connections (α_0, β_0) and (α_1, β_1)

$$CQPM(M) \quad \underset{\beta_2 = \beta_0 \circ \beta_1}{\overset{\alpha_2 = \alpha_1 \circ \alpha_0}{\rightleftarrows}} \quad L - SB(M) \ .$$

A direct computation of α_2 and β_2 gives

$$\alpha_2(d)(x) = h^{-1}(d(e, x))$$

and

$$\beta_2(m)(x, y) = \inf(\{h(m(t)) \mid t \cdot x = y\}).$$

It is also immediate to prove that

$$\beta_2(\alpha_2(d)) \preceq d \ \ ; \ \ \alpha_2(\beta_2(m)) = m.$$

Definition 26. We denote by (α'_2, β'_2) the Galois connection from $QPM(S)$ to $L - SM(S^S)$ obtained by composing the Galois connections (α_0, β_0) and (α'_1, β'_1)

$$QPM(S) \quad \underset{\beta'_2 = \beta_0 \circ \beta'_1}{\overset{\alpha'_2 = \alpha'_1 \circ \alpha_0}{\rightleftarrows}} \quad L - SM(S^S) \ .$$

A direct computation of α'_2 gives

$$\alpha'_2(d)(f) = h^{-1}(d'(e, f))$$

where $d' : S^S \to [0, +\infty]$ is the compatible extended quasi pseudo metric defined by setting, for any $f, g \in S^S$,

$$d'(f, g) = \sup_{x \in S} d(f(x), g(x)). \tag{17}$$

Moreover β'_2 is defined by

$$\beta'_2(m)(x, y) = \inf(\{h(m(f)) | f(x) = y\}).$$

It is immediate to prove that

$$\beta'_2(\alpha'_2(d)) = d \ \ ; \ \ \alpha'_2(\beta'_2(m)) \supseteq m.$$

7 Similarities, L-subgroups and distances

The notion of L-subgroup is defined as a many valued extension of the classical one of subgroup (as an example see [2] and [21]).

Definition 27. Let $(G, \cdot, ^{-1}, e)$ be a group. Then an *L-subgroup* of G is an L-submonoid g of G such that $g(x^{-1}) \geq g(x)$ for every $x \in G$.

To obtain a formula for the L-subgroup generated by an L-subset, we denote by s^{-1} the L-subset of G defined by setting $s^{-1}(x) = s(x^{-1})$. The proof of the following proposition is immediate.

Proposition 28. *The class $L - SG(G)$ of L-subgroups of G is a closure system and therefore a complete lattice. Given an L-subset s of G, let $\overline{s}$ be the L-subset defined by setting $\overline{s}(e) = 1$ and, in the case $x \neq e$*

$$\overline{s}(x) = \sup\{\widetilde{s}(x_1) * ... * \widetilde{s}(x_n) : x_1 \cdot ... \cdot x_n = x\}$$

where $\widetilde{s} = s \cup s^{-1}$. Then $\overline{s}$ is the L-subgroup generated by s.

The symmetric L-preorders define the important class of L-similarities.

Definition 29. An *L-similarity,* in brief *similarity,* on S is an L-preorder relation r satisfying the symmetric property, i.e.

iii) $r(x, y) = r(y, x)$ for every $x, y \in S$.

The notion of similarity extends the classical one of equivalence relation. To obtain the similarity generated by a given L-relation, in the following, given an L-relation s in S, we denote by s^{-1} the L-relation defined by setting $s^{-1}(x, y) = s(y, x)$. The proof of the following proposition is immediate (see [1]).

Proposition 30. *The class $L - SI(S)$ of all the L-similarities on S is a closure system and therefore a complete lattice. Given an L-relation $s : S \times S \to L$, let $\overline{s}$ be the L-relation defined by setting $\overline{s}(x, x) = 1$ and, in the case $x \neq y$,*

$$\overline{s}(x, y) = \sup\{\widetilde{s}(x_1, x_2) * ... * \widetilde{s}(x_{n-1}, x_n) : x_1 = x, x_n = y\},$$

where $\widetilde{s} = s \cup s^{-1}$. Then $\overline{s}$ is the similarity generated by s.

Observe that a compatible L-relation r in a group G satisfies the identity

$$r(x, y) = r(x \cdot z, y \cdot z).$$

Also, for any L-subset g of G,

$$\beta_1(g)(x, y) = g(y \cdot x^{-1}).$$

This means that $\beta_1(g)$ coincides with the L-relation defined in [19] where the following proposition was also proved.

Proposition 31. *Let G be a group and r be a compatible similarity in G. Then $\alpha_1(r)$ is an L-subgroup of G. Let g be an L-subgroup of G, then $\beta_1(g)$ is a compatible similarity. Moreover,*

$$\beta_1(\alpha_1(r)) = r \quad ; \quad \alpha_1(\beta_1(g)) = g. \tag{18}$$

Then (α_1, β_1) is a Galois connection, namely a lattice isomorphism, from the lattice $L - CSI(G)$ of the compatible similarities in G into the lattice $L - SG(G)$ of all the L-subgroups of G

$$L - CSI(G) \quad \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \quad L - SG(G) \ .$$

In particular, we are interested to the group Σ_S of transformations on the set S. The following theorem was proved in [8] and [13].

Theorem 32. *Let r be a similarity in S. Then $\alpha'_1(r)$ is an L-subgroup of Σ_S. Let g be an L-subgroup of Σ_S, then $\beta'_1(g)$ is a similarity in S. Then (α'_1, β'_1) is a Galois connection from $L - SI(S)$ to $L - SG(\Sigma_S)$.*

$$L - SI(S) \quad \underset{\beta'_1}{\overset{\alpha'_1}{\rightleftarrows}} \quad L - SG(\Sigma_S)$$

Proof. See [12]. ■

Assume that L is the residuated lattice defined in $[0,1]$ by a t-norm $*$ generated by a function h such that $h(0) = \infty$. Also, denote by $PM(S)$ the lattice of the extended pseudo-metric spaces and by $CPM(M)$ the lattice of compatible extended pseudo-metric spaces. Then we can list some theorems whose proofs are obvious.

Theorem 33. *If d is an extended pseudo-metric then $\alpha_0(d)$ is a L-similarity. If r is an L-similarity, then $\beta_0(r)$ is an extended pseudo-metric. As a consequence, the pair (α_0, β_0) is a Galois connection from the lattice $PM(S)$ of the extended pseudo-metric spaces into the lattice $L-SI(S)$ of the L-similarities in S*

$$PM(S) \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} L - SI(S)\ .$$

Theorem 34. *Let G be a group. Then if d is a compatible pseudo-metric in G, then $\alpha_0(d)$ is a compatible L-similarity in G. If r is a compatible L-similarity in G, then $\beta_0(r)$ is a compatible pseudo-metric. As a consequence, the pair (α_0, β_0) is a Galois connection from the lattice $CPM(G)$ of compatible pseudo-metrics in G into the lattice $L - CSI(G)$ of the compatible L-similarities in G*

$$CPM(G) \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} L - CSI(G)\ .$$

Theorem 35. *The pair (α_2, β_2) is a Galois connection from $CPM(G)$ to $L - SG(G)$*

$$CPM(G) \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} L - SG(G)\ .$$

Theorem 36. *The pair (α_2', β_2') is a Galois connection from $PM(S)$ to $L - SG(S^S)$*

$$PM(S) \underset{\beta_2'}{\overset{\alpha_2'}{\rightleftarrows}} L - SG(S^S)\ .$$

8 Examples.

The results in this papers enable us to find several examples of fuzzy submonoids and fuzzy subgroups (and therefore of fuzzy orders, quasi-metrics, similarities and distances).

Fuzzy code theory Let A be an alphabet and denote by A^* the related free monoid. An important class of submonoids of A^* are the *free* submonoids, i.e. the submonoids M such that

$$xy \in M, yx \in M, y \in M \Rightarrow x \in M.$$

The interest of these submonoids is that the set $M - M^2$ of words is a code (see, for example [17]). Further classes of codes are obtained by considering the *pure*, *very pure*, *left unitary* submonoids, i.e. the submonoids M satisfying the following implications:

- $x^n \in M \Rightarrow x \in M$,
- $x \cdot y \in M, y \cdot x \in M \Rightarrow x \in M$,
- $y \cdot x \in M, y \in M \Rightarrow x \in M$,

respectively. In accordance with [9], and [13] we extend these definitions as follows.

Definition 37. An L-submonoid m of A^* is called *free*, *pure*, *very pure*, *left unitary*, provided that

a) $m(x) \geq m(x \cdot y) * m(y \cdot x) * m(y)$,

b) $m(x^n) = m(x)$,

c) $m(x) \geq m(x \cdot y) * m(y \cdot x)$,

d) $m(x) \geq m(y \cdot x) * m(y)$,

respectively.

In the case $*$ is the meet operation $\wedge$, m is a free L-submonoid if and only if for any $\lambda \in L$, the cut $C(m, \lambda) = \{x \in A^* : m(x) \geq \lambda\}$ is a free submonoid. So, in such a case, any free L-submonoid gives a whole family of codes depending on the parameter λ. Analogous considerations hold true for the pure, very pure, left unitary L-submonoids.

In [10] the following proposition was proved.

Proposition 38. *The class of free (pure, very pure, left unitary) L-submonoids of A^* is a closure system.*

In [10] and [9] we proposed some formulas for the free (pure, very pure, left unitary) L-submonoid generated by a given L-subset.

The just considered L-submonoids of A^* are related with suitable of L-preorders and quasi-metrics, obviously. For example, we say that a compatible L-preorder r in A^* is *free*, *pure*, *very pure*, *left unitary*, provided that

a) $r(e, x) \geq r(e, x \cdot y) * r(e, y \cdot x) * r(y)$,

b) $r(e, x^n) = r(e, x)$,

c) $r(e, x) \geq r(e, x \cdot y) * r(e, y \cdot x)$,

d) $r(e, x) \geq r(e, y \cdot x) * r(e, y)$,

respectively. In accordance with the results in this paper we obtain the following proposition.

Proposition 39. *The pair (α_1, β_1) is a Galois connection from the lattice of the compatible free (pure, very pure, left unitary) L-preorders into the lattice of the free (pure, very pure, left unitary) L-submonoids of A^*.*

We conclude this section by observing that we can define in a simple way the L-preorder associated with a given L-submonoid of S^*.

Proposition 40. *Define the partial operation y/x in A^* by setting $y/x = z$ if x is a right factor of y and $z \cdot x = y$. Then the fuzzy preorder $\beta_1(m)$ m be an L-submonoid of A^*.can be obtained by setting*

$$\beta_1(m)(x, y) = m(y/x)$$

if x is a right factor of y and $\beta_1(m)(x, y) = 0$ otherwise. Moreover, for any $z \in A^*$,

$$\beta_1(m)(x, y) = \beta_1(m)(xz, yz).$$

The L-submonoid of the easy to compute functions. Further examples of fuzzy submonoids and therefore of fuzzy preorders are suggested by Kolmogorov's complexity theory (see, for example [18]). As an example, let $Comp$ the set of one-variable computable functions from N to N and consider the monoid $(Comp, \circ, i)$. Also, for any $f \in Comp$, let $l(f)$ be the length of a shortest program to compute f in an universal programming language. More precisely, we refer to programs starting with an input instruction like *Read(x)* and terminating with an output instruction *Write(y)* and we do not consider these two instruction in computing the length of a program.

Proposition 41. *Assume that $L = [0, 1]$, let h be an additive generator and define $m : Comp \to [0, 1]$ by setting*

$$m(f) = h^{[-1]}(l(f)).$$

Then m is an L-submonoid of $Comp$ with respect to the t-norm generated by h.

Proof. See [12]. ∎

We interpret m as the L-monoid of the easy to compute functions. Such an L-submonoid is associated with an L-preorder $r = \beta_1(m)$ defined by

$$r(f, g) = \sup\{m(t) : t \circ f = g\}.$$

We interpret the value $r(f, g)$ as a valuation in a multivalued logic of the claim "there is an easy to calculate function able to obtain g from f by composition". Also, assume that $L = [0, 1]$ and that $h(0) = \infty$. Then we can obtain the quasi-metric $d = \beta_2(m)$ associated with m. Obviously,

$$d(f, g) = \inf\{l(t) : t \circ f = g\}.$$

Transformation distances and DNA evolution. In literature there are examples of quasi-metrics which are related with the question of the evolution. Indeed, evolution acts in several ways on DNA : either by mutating a base, or inserting, deleting or copying a segment of the given sequence. So, the general situation is that if we denote by S the set of DNA-words, then a set of possible operations in S is fixed. Once a cost is fixed to each of these operations, the total cost of a list of operations (a script) is calculated simply by adding the costs of the single operations (see [24]). Under these

hypotheses, given an initial DNA-word x and a target DNA-word y, a non symmetric distance $d(x, y)$ from x to y is defined as the minimum cost among all the scripts able to generate y from x. Such a notion was suggested by algorithmic information theory.
Now, we can translate such an approach in terms of generated L-submonoids and L-orders. Indeed, let s be an L-subset of S^S we interpret as an L-subset of possible transformations. Then, we can consider the L-submonoid $\overline{s}$ of S^S generated by s and therefore, in accordance with (3), the L-preorder $\beta_1(\overline{s})$ defined by setting

$$\beta_1(\overline{s})(x, y) = \sup\{s(f_1) * ... * s(f_n) : f_1(...f_n(x)...) = y\}.$$

The quasi metric d defined in [24] is obtained as a particular case by assuming that $L = [0, 1]$ and by considering $\beta_0(\beta_1(\overline{s}))$.

Obviously it is not clear if such an approach is more convenient than the usual one of the transformation distances. A reason in favour of the L-submonoids (equivalently, the L-preorders) is the remarkable generality of such a notion. Indeed, we can start from residuated lattices L which are not necessarily based on the set of real numbers.

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