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POINTLESS METRIC SPACES

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§1. Introduction. In the last years several research projects have been motivated by the problem of constructing the usual geometrical spaces by admitting "regions" and "inclusion" between regions as primitives and by defining the points as suitable sequences or classes of regions (for references see [2]).

In this paper we propose and examine a system of axioms for the pointless space theory in which "regions", "inclusion", "distance" and "diameter" are assumed as primitives and the concept of point is derived. Such a system extends a system proposed by K. Weihrauch and U. Schreiber in [5].

In the sequel **R** and **N** denote the set of real numbers and the set of natural numbers, and E is a Euclidean metric space. Moreover, if X is a subset of **R**, then $\bigvee X$ is the least upper bound and $\bigwedge X$ the greatest lower bound of X.

§2. Pointless metric spaces. By a *pointless pseudometric space*, briefly p-p-m-space, we mean any structure $\mathscr{R} = (R, \leq, \delta, |\ |)$ where $(R, \leq)$ is a partially ordered set and $|\ |: R \to [0, \infty]$, $\delta: R \times R \to [0, \infty)$ are functions such that, for every $x, y, z \in R$, the following axioms hold:

A1. $x \geq y \Rightarrow |x| \geq |y|$.

A2. $x \geq y \Rightarrow \delta(y, z) \geq \delta(z, x)$.

A3. $\delta(x, x) = 0$.

A4. $\delta(x, y) \leq \delta(x, z) + \delta(z, y) + |z|$ (generalized triangle inequality).

A similar system of axioms was defined by Weihrauch and Schreiber in [5]. We call the elements of R *regions*, the relation $\leq$ *inclusion*, the number $\delta(x, y)$ the *distance* between the regions x and y and the number $|x|$ the *diameter* of x. We say that an element 0 of R is *the empty region* if it is the minimum of $(R, \leq)$, and we denote by R_0 the set of nonempty regions. Notice that if the empty region 0 exists then δ is constantly equal to zero. Indeed, by A2 $\delta(0, 0) \geq \delta(0, x)$ and therefore, by A3, $\delta(0, x) = 0$. Consequently, since by A2 $\delta(z, x) \leq \delta(0, z)$, we have also that $\delta(z, x) = 0$.

The minimal elements of R_0 are called *atoms* while the elements of R_0 whose diameter is zero are called *$\mathscr{R}$-points*. A region x is *bounded* if $|x| \neq \infty$; two regions x

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and y *overlap* if $z \in R_0$ exists such that $z \leq x$ and $z \leq y$. A *homomorphism* from a p-p-m-space $\mathscr{R} = (R, \leq, \delta, |\ |)$ into a p-p-m-space $\mathscr{R}' = (R', \leq', \delta', |\ |')$ is a map from R into R' preserving order, distances and diameters; an *isomorphism* is an injective and surjective homomorphism. Notice that, as for the pseudometric spaces, every (nonempty) subset of a p-p-m-space defines a p-p-m-space.

The p-p-m-spaces generalize the pseudometric spaces; namely, the pseudometric spaces coincide with the p-p-m-spaces for which every region is an atom whose diameter is zero, that is the order relation coincides with the identity relation and $|\ |$ is constantly equal to zero.

Examples of p-p-m-spaces are obtained when R is equal to a class of nonempty subsets of a pseudometric space (M, d), $\leq$ is the inclusion relation and δ and $|\ |$ are the usual distance and diameter functions defined by

$$(1)\quad \delta(X, Y) = \bigwedge\{d(x, y) \mid x \in X, y \in Y\}; \qquad |X| = \bigvee\{d(x, y) \mid x \in X, y \in X\}.$$

Indeed A1, A2 and A3 are immediate; to prove A4, let X, Y and Z be subsets of M, $x \in X, y \in Y, z \in Z$ and $z' \in Z$; then

$$\delta(X, Y) \leq d(x, y) \leq d(x, z) + d(z, z') + d(z', y) \leq d(x, z) + d(z', y) + |Z|$$

and therefore $\delta(X, Y) \leq \delta(X, Z) + \delta(Z, Y) + |Z|$. We call the spaces of this type *canonical*.

The passage from the pointless pseudometric spaces to the pointless metric spaces is suggested by the following observations. We recall that a metric space is a pseudometric space such that $x = y \Leftrightarrow d(x, y) = 0$; in other words the identity relation can be defined via the distance function. This suggests calling a *pointless metric space*, briefly p-m-space, a p-p-m-space $\mathscr{R} = (R, \leq, \delta, |\ |)$ such that

A5. $\quad x \geq y \Leftrightarrow |x| \geq |y|$ and $\delta(z, x) \leq \delta(y, z)$ for every $z \in R$.

Obviously A5 is equivalent to A1, A2 and the implication

$$|x| \geq |y| \text{ and } \delta(z, x) \leq \delta(y, z) \text{ for every } z \in R \Rightarrow x \geq y.$$

The metric spaces coincide with the p-m-spaces such that the regions are atoms whose diameter is equal to zero. Indeed, in this case if $\delta(x, y) = 0$ then $\delta(z, x) \leq \delta(z, y) + \delta(y, x) + |y| = \delta(y, z)$ for every $z \in R$, and therefore $x \geq y$. Likewise we prove that $y \geq x$ and therefore $x = y$. A canonical space in a metric space is not a p-m-space, in general. For example, if R is the class of the open and closed balls of a Euclidean metric space, y is an open ball and x its closure, then A5 does not hold. By confining ourselves to the class of open balls, we obtain a p-m-space.

The following proposition shows some properties of the p-p-m-spaces.

PROPOSITION 1. *In any p-p-m-space the following hold*:

$$(2)\qquad \delta(x, y) = \delta(y, x),$$

$$(3)\qquad x \geq y \Rightarrow \delta(y, x) = 0,$$

$$(4)\qquad x \text{ and } y \text{ overlap} \Rightarrow \delta(x, y) = 0,$$

$$(5)\quad |z| \geq \bigvee\{\delta(x, y) - \delta(x, z) - \delta(z, y) \mid x, y \in R\} \geq \bigvee\{\delta(x, y) \mid x \leq z \text{ and } y \leq z\},$$

(6) $\mathscr{R}$ *a p-m-space* $\Rightarrow$ *every* $\mathscr{R}$*-point is an atom.*

PROOF. By setting $y = x$ in A2 we obtain that $\delta(x,z) \geq \delta(z,x)$ and therefore that $\delta(x,z) = \delta(z,x)$. To prove (3), set $z = y$ in A2; then by A3 we have $\delta(y,x) \leq \delta(y,y) = 0$. To prove (4), assume that $z \leq x$ and $z \leq y$; then $\delta(x,y) \leq \delta(x,z) \leq 0$. (5) is an immediate consequence of A4. To prove (6), observe that if y is an $\mathscr{R}$-point and $x \leq y$, then, for every $z \in R$,

$$\delta(z,x) \leq \delta(z,y) + \delta(y,x) + |y| = \delta(y,z).$$

Since $|x| = |y| = 0$, by A5 we have $x \geq y$. Thus $x = y$, and therefore y is an atom. □

§3. Diameter, distance and inclusion relation as derived concepts. In defining the p-p-m-spaces we have assumed as primitives $\leq$, δ and $|\ |$, but it is also possible to assume as primitives only two of these notions by defining the remaining one.

In order to show this, we have to give some definitions. Let $(R, \leq)$ be an ordered set and $|\ |: R \to [0,\infty]$ a function; then if $x, y \in R$, we call any finite sequence $b_1,\dots,b_n$ of bounded regions such that x overlaps b_1, b_i overlaps b_{i+1}, for $i = 1,\dots,n-1$ and b_n overlaps y a *path connecting* x *with* y. We define the function $\delta_{|\ |}: R \times R \to [0,\infty]$ by

$$(7)\quad \delta_{|\ |}(x,y) = \begin{cases} 0 & \text{if } x \text{ overlaps } y, \\ \bigwedge\{|b_1| + \cdots + |b_n| \mid b_1,\dots,b_n \text{ is a path from } x \text{ to } y\} & \text{otherwise.} \end{cases}$$

The definition of $\delta_{|\ |}$ was given in [5]. Moreover, for every function $\delta: R \times R \to [0,\infty)$ we set

$$(8)\qquad |z|_\delta = \bigvee\{\delta(x,y) - \delta(x,z) - \delta(z,y) \mid x, y \in R\}.$$

PROPOSITION 2. *Let* $(R, \leq)$ *be an ordered set and* $|\ |: R \to [0,\infty)$ *a function satisfying* A1; *and assume that two regions are always connected by a path. Then* $(R, \leq, \delta_{|\ |}, |\ |)$ *is a p-p-m-space. If* $\delta: R \times R \to [0,\infty)$ *is a function satisfying* A2 *and* A3, *then* $(R, \leq, \delta, |\ |_\delta)$ *is a p-p-m-space.*

PROOF. Axioms A2 and A3 are immediate consequences of the definition of $\delta_{|\ |}$. To prove A4, assume that z does not overlap x nor y. Then

$$\begin{aligned} &\delta_{|\ |}(x,z) + \delta_{|\ |}(z,y) + |z| \\ &\quad = \bigwedge\{|b_1| + \cdots + |b_n| \mid b_1,\dots,b_n \text{ is a path from } x \text{ to } z\} \\ &\qquad + \bigwedge\{|b_1| + \cdots + |b_n| \mid b_1,\dots,b_n \text{ is a path from } z \text{ to } y\} + |z| \\ &\quad \geq \bigwedge\{|b_1| + \cdots + |b_n| \mid b_1,\dots,b_n \text{ is a path from } \\ &\qquad\qquad x \text{ to } y \text{ and } b_i = z \text{ for a suitable } i\} \\ &\quad \geq \delta_{|\ |}(x,y). \end{aligned}$$

One proceeds similarly in the case that z overlaps x, y or both. The rest of the proposition is obvious. □

PROPOSITION 3. *Let* $(R, \leq, \delta, |\ |)$ *be a p-p-m-space. Then* $(R, \leq, \delta, |\ |_\delta)$ *is a p-p-m-space such that* $|\ |_\delta \leq |\ |$; *namely,* $|\ |_\delta$ *is the smallest diameter compatible with* δ. *If in* $(R, \leq, \delta, |\ |)$ *two regions are always connected by a path, then* $(R, \leq, \delta_{|\ |}, |\ |)$ *is a p-p-m-space such that* $\delta_{|\ |} \geq \delta$; *namely,* $\delta_{|\ |}$ *is the largest distance compatible with* $|\ |$.

PROOF. We limit ourselves to proving that $\delta(x, y) \leq \delta_{|\ |}(x, y)$. Now, if $b_1, \ldots, b_n$ is a path from x to y, then by A4 we have

$$\delta(x, y) \leq \delta(x, b_1) + \sum_i \delta(b_i, b_{i+1}) + \delta(b_n, y) + \sum_i |b_i| = \sum_i |b_i|;$$

hence $\delta(x, y) \leq \delta_{|\ |}(x, y)$. □

Notice that if $\mathscr{R}$ is a p-p-m-space then $\delta \neq \delta_{|\ |}$ and $|\ | \neq |\ |_\delta$, in general. This is due to the very weak connection between δ and $|\ |$ imposed by A4. For example, if $\mathscr{R} = (R, \leq, \delta, |\ |)$ is a p-p-m-space, $\alpha \in [0, 1]$, $\beta \in [1, \infty)$ and $\gamma \geq 0$, then $(R, \leq, \alpha \cdot \delta, \beta \cdot |\ | + \gamma)$ is a p-p-m-space.

The definition of $\leq$ via δ and $|\ |$ is a bit more complicated. Let R be a set equipped with two functions δ: $R \times R \to [0, \infty)$ and $|\ |$: $R \to [0, \infty]$. Then we set

$$x \equiv y \Leftrightarrow |x| = |y| \text{ and } \delta(y, z) = \delta(x, z) \text{ for every } z \in R;$$

$$[x] = \{y \in R \mid y \equiv x\}; \quad R^* = \{[x] \mid x \in R\}; \quad \delta^*([x], [y]) = \delta(x, y); \quad |[x]|^* = |x|;$$

$$[x] \leq^* [y] \Leftrightarrow |x| \leq |y| \text{ and } \delta(y, z) \leq \delta(x, z) \text{ for every } z \in R.$$

PROPOSITION 4. *If R is a set and δ: $R \times R \to [0, \infty)$ and $|\ |$: $R \to [0, \infty]$ are two functions such that*

(i) $\delta(x, x) = 0$, (ii) $\delta(x, y) = \delta(y, x)$, (iii) $\delta(x, y) \leq \delta(x, z) + \delta(z, y) + |x|$,

then $(R^*, \leq^*, \delta^*, |\ |^*)$ is a p-m-space. Consequently, if $(R, \leq, \delta, |\ |)$ is a p-p-m-space, then $\mathscr{R}^* = (R^*, \leq^*, \delta^*, |\ |^*)$ is a p-m-space such that

$$x \leq y \Rightarrow [x] \leq^* [y]. \tag{9}$$

PROOF. Obvious. □

The p-m-space $\mathscr{R}^*$ in Proposition 4 is called the *quotient* of $\mathscr{R}$; this extends to the p-p-m-spaces the well-known notion of quotient of a pseudometric space.

Propositions 2 and 4 show that it is very easy to build up examples of p-p-m-spaces. For example, if R is a set and f: $R \times R \to [0, \infty)$ is any function, then by setting $\delta(x, y) = 0$ if $x = y$ and $\delta(x, y) = (f(x, y) + f(y, x))/2$ otherwise, we obtain a function satisfying (i) and (ii). Since $|\ |_\delta$ and δ satisfy (iii), by Proposition 4 we obtain a p-m-space.

§4. The points of a pointless pseudometric space. To define the points in a p-p-m-space, we utilize a procedure similar to the completion of a metric space via Cauchy sequences. By a *Cauchy sequence* of a p-p-m-space $\mathscr{R}$ we mean any sequence $\langle p_n \rangle$ of nonempty bounded regions such that

a) $\lim |p_n| = 0$, and

b) $\forall \varepsilon > 0\ \exists \nu\ \forall h \geq \nu\ \forall k \geq \nu\ \delta(p_h, p_k) < \varepsilon$.

Decreasing sequences of nonempty regions with vanishing diameters are examples of Cauchy sequences. Obviously, it is possible that in a p-p-m-space there is no Cauchy sequence.

PROPOSITION 5. *Assume that the class $S(\mathscr{R})$ of the Cauchy sequences of $\mathscr{R}$ is nonempty, and define d: $S(\mathscr{R}) \times S(\mathscr{R}) \to [0, \infty)$ by*

$$d(\langle p_n \rangle, \langle q_n \rangle) = \lim \delta(p_n, q_n) \quad \textit{for every } \langle p_n \rangle \in S(\mathscr{R}) \textit{ and } \langle q_n \rangle \in S(\mathscr{R}). \tag{10}$$

Then $(S(\mathscr{R}), d)$ *is a pseudometric space.*

PROOF. First we have to prove that the sequence $\langle \delta(p_n, q_n) \rangle$ is convergent, i.e.

$$\forall \varepsilon > 0\ \exists \nu\ \forall m \geq \nu\ |\delta(p_\nu, q_\nu) - \delta(p_m, q_m)| < \varepsilon. \tag{11}$$

Now, from $\delta(p_\nu, q_\nu) \leq \delta(p_\nu, p_m) + \delta(p_m, q_m) + \delta(q_m, q_\nu) + |p_m| + |q_m|$ it follows that

$$\delta(p_\nu, q_\nu) - \delta(p_m, q_m) \leq \delta(p_\nu, p_m) + \delta(q_m, q_\nu) + |p_m| + |q_m|.$$

Since

$$\forall \varepsilon > 0\ \exists \nu_1\ \forall m \geq \nu_1\ |p_m| \leq \varepsilon/4,$$
$$\forall \varepsilon > 0\ \exists \nu_2\ \forall m \geq \nu_2\ |q_m| \leq \varepsilon/4,$$
$$\forall \varepsilon > 0\ \exists \nu_3\ \forall m \geq \nu_3\ \forall \nu \geq \nu_3\ \delta(p_\nu, p_m) \leq \varepsilon/4,$$
$$\forall \varepsilon > 0\ \exists \nu_4\ \forall m \geq \nu_4\ \forall \nu \geq \nu_4\ \delta(q_\nu, q_m) \leq \varepsilon/4,$$

by setting $\nu' = \max\{\nu_1, \nu_2, \nu_3, \nu_4\}$ we have that, for every $m \geq \nu'$, $\delta(p_\nu, q_\nu) - \delta(p_m, q_m) < \varepsilon$. Likewise, since

$$\delta(p_m, q_m) - \delta(p_\nu, q_\nu) \leq \delta(p_m, p_\nu) + \delta(q_\nu, q_m) + |p_\nu| + |q_\nu|,$$

an integer ν'' exists such that, for every $m \geq \nu''$, $\delta(p_m, q_m) - \delta(p_\nu, q_\nu) < \varepsilon$. By setting $\nu = \max\{\nu', \nu''\}$, we obtain (11).

To prove that $(S(\mathscr{R}), d)$ is a pseudometric space, we limit ourselves to proving the triangle inequality. Now, if $\langle p_n \rangle$, $\langle q_n \rangle$ and $\langle z_n \rangle$ are elements of $S(\mathscr{R})$, then

$$\begin{aligned} d(\langle p_n \rangle, \langle q_n \rangle) &= \lim \delta(p_n, q_n) \leq \lim(\delta(p_n, z_n) + \delta(z_n, q_n) + |z_n|) \\ &= \lim \delta(p_n, z_n) + \lim \delta(z_n, q_n) + \lim |z_n| \\ &= d(\langle p_n \rangle, \langle z_n \rangle) + d(\langle z_n \rangle, \langle q_n \rangle). \end{aligned}$$ □

We denote by $(M(\mathscr{R}), d)$ the metric space obtained as a quotient of $(S(\mathscr{R}), d)$ modulo the relation $\equiv$ defined by setting $\langle p_n \rangle \equiv \langle q_n \rangle$ if $d(\langle p_n \rangle, \langle q_n \rangle) = 0$. Moreover each element of $M(\mathscr{R})$ is called a *point*; as a consequence, a point p is a class

$$[\langle p_n \rangle] = \{\langle q_n \rangle \in S(\mathscr{R}) \mid \langle q_n \rangle \equiv \langle p_n \rangle\},$$

and $d\colon M(\mathscr{R}) \times M(\mathscr{R}) \to [0, \infty)$ is defined by setting, for every $p, q \in M(\mathscr{R})$,

$$d(p, q) = d(\langle p_n \rangle, \langle q_n \rangle) = \lim \delta(p_n, q_n), \tag{12}$$

where $\langle p_n \rangle \in S(\mathscr{R})$ and $\langle q_n \rangle \in S(\mathscr{R})$ represent p and q, respectively. Notice that a p-p-m-space $\mathscr{R}$ and its quotient $\mathscr{R}^*$ determine the same metric space.

To every $\mathscr{R}$-point r we can associate the point represented by the sequence constantly equal to r. In a p-m-space different $\mathscr{R}$-points represent different elements of $M(\mathscr{R})$.

If $\mathscr{R}$ is the p-m-space with zero diameters associated to a metric space (M, d), then $(M(\mathscr{R}), d)$ is the completion of (M, d), obviously. If $\mathscr{R}$ is the canonical p-m-space of the open balls of a Euclidean space E, then $(M(\mathscr{R}), d)$ coincides with E.

If $p \in M(\mathscr{R})$ and $r \in R$, we say that *p belongs to r*, in brief $p \,\varepsilon\, r$, provided that there is a sequence $\langle p_n \rangle$ representing p with $p_n \leq r$ for every $n \in \mathbf{N}$. Let $P(r)$ denote the set of points belonging to r.

Notice that these definitions are different from the analogous ones given in [1] and [5]. We examine them in §6.

It is also possible to define the distance between a point and a region by

$$d(p,r) = d(r,p) = \lim \delta(p_n, r), \qquad p \in M(\mathscr{R}),\ r \in R_0. \tag{13}$$

This definition does not depend on the sequence $\langle p_n \rangle$ representing p; moreover, for every $p, q \in M(\mathscr{R})$ and $r, s \in R_0$, the following hold:

$$p \,\varepsilon\, r \Rightarrow d(p,r) = 0, \tag{14}$$

$$d(p,r) \leq d(p,q) + d(q,r), \tag{15}$$

$$d(p,q) \leq d(p,r) + d(r,q) + |r|, \tag{16}$$

$$d(r,s) \leq d(r,p) + d(p,s). \tag{17}$$

PROPOSITION 6. *If $S(\mathscr{R})$ is nonempty, then $(M(\mathscr{R}), d)$ is a complete metric space and, for every $r \in R$, $P(r)$ is a closed subset. Moreover, for every $r, s \in R$,*

$$r \leq s \Rightarrow P(r) \subseteq P(s), \tag{18}$$

and, if $P(r)$ and $P(s)$ are nonempty,

$$\delta(r,s) \leq \delta(P(r), P(s)), \tag{19}$$

$$|P(r)| \leq |r|_\delta \leq |r|. \tag{20}$$

PROOF. To prove the completeness, we first observe that

$$\forall p \in M(\mathscr{R})\ \forall \varepsilon > 0\ \exists s \in R\ |s| < \varepsilon \text{ and } d(p,s) < \varepsilon. \tag{21}$$

Moreover, if $p \in P(r)$ then $s \leq r$. Indeed, if p is represented by $\langle p_n \rangle$, then for every $\varepsilon > 0$ there exists ν such that $\delta(p_\nu, p_n) < \varepsilon$ and $|p_\nu| < \varepsilon$ for every $n \geq \nu$. Since $d(p_\nu, p) = \lim \delta(p_\nu, p_n) < \varepsilon$, by setting $s = p_\nu$ we obtain (21).

Now, let $\langle p^n \rangle$ be a Cauchy sequence of elements of $M(\mathscr{R})$, and let s_n be a region such that $\delta(s_n, p^n) < 1/n$ and $|s_n| < 1/n$. Then

$$\delta(s_n, s_m) \leq \delta(s_n, p^n) + d(p^n, p^m) + d(p^m, s_m) < 1/n + d(p^n, p^m) + 1/m$$

and $\langle s_n \rangle$ is a Cauchy sequence of $\mathscr{R}$ representing a point p. Since

$$d(p, p^n) \leq d(p, s_n) + d(s_n, p^n) + |s_n| \leq d(p, s_n) + 1/n + 1/n,$$

we have that $\lim d(p, p^n) = 0$.

To prove that $P(r)$ is closed, observe that if in the above proof we assume that $p^n \in P(r)$ for every $n \in \mathbf{N}$, then $s_n \leq r$ and therefore $p \in P(r)$.

(18) is obvious. To prove (19), observe that, for every $p \in P(r)$ and $q \in P(s)$,

$$\delta(r,s) \leq \delta(r,p) + d(p,q) + \delta(q,s) \leq d(p,q).$$

To prove (20), observe that, if $p, q \in P(r)$, since $\delta(p_n, r) = \delta(r, q_n) = 0$, then

$$d(p,q) \leq \bigvee \delta(p_n, q_n) \leq \bigvee \{\delta(x,y) - \delta(x,r) - \delta(r,y) \mid x, y\} = |r|_\delta. \qquad \square$$

§5. Existence of points. Axioms A1–A5 do not assure the existence of points in a p-p-m-space. To obtain this, we have to add some new axiom. For example, we can assume that every nonempty region contains arbitrarily small nonempty regions:

A6. $\forall\varepsilon > 0\ \forall x \in R_0\ \exists x' \in R_0\ x' \leq x, |x'| < \varepsilon.$

Obviously, A6 is equivalent to saying that every nonempty region r has points, i.e. $P(r) \neq \emptyset$ for every $r \in R_0$. As a consequence, if R_0 is nonempty, the class $P(\mathscr{R}) = \{P(r) \subseteq M(\mathscr{R}) \mid r \in R_0\}$ of subsets of $M(\mathscr{R})$ defines a canonical p-p-m-space $\mathscr{R}'$, the *canonical p-p-m-space associated to* $\mathscr{R}$.

From A6 it follows also that every atom of $(R, \leq)$ is an $\mathscr{R}$-point. Then in the p-m-spaces satisfying A6 the $\mathscr{R}$-points coincide with the atoms.

It is very natural to investigate the relationship between a p-p-m-space and its associated canonical p-p-m-space.

PROPOSITION 7. *Assume that the p-p-m-space* $\mathscr{R}$ *satisfies* A6. *Then*

(22) $\delta(r,s) \leq \delta(P(r),P(s)) \leq \delta_{|\,|}(r,s)$ *and* $|P(r)| = \bigvee\{\delta(u,v)/u \leq r, v \leq r\}$

for every pair of nonempty regions r and s.

PROOF. If r and s overlap, then $\delta_{|\,|}(r,s) = 0$, and, since $P(r) \cap P(s) \neq \emptyset$, we have also that $\delta(P(r),P(s)) = 0$. If r and s do not overlap, let $r_1,\ldots,r_n$ be any path from r to s and let $p_1,\ldots,p_{n+1}$ be a sequence of points such that $p_1 \in P(r) \cap P(r_1)$, $p_{n+1} \in P(r_n) \cap P(s)$, and $p_i \in P(r_{i-1}) \cap P(r_i)$ for $i = 2,\ldots,n$. Then

$$\delta(P(r), P(s)) \leq d(p_1,p_{n+1}) \leq \sum_i d(p_i,p_{i+1}) \leq \sum_i |r_i|.$$

This proves that $\delta(P(r),P(s)) \leq d_{|\,|}(r,s)$. Since (19) holds, the first part of (22) is proved.

Let $u \leq r$ and $v \leq r$ and, by A6, let p and q be two points of u and v, respectively. Then $\delta(u,v) \leq d(p,q) \leq |P(r)|$ and this proves that

$$\bigvee\{\delta(u,v) \mid u \leq r, v \leq r\} \leq |P(r)|.$$

Conversely, let p and q be two points of r; then

$$d(p,q) = \bigvee\delta(p_n,q_n) \leq \bigvee\{\delta(u,v) \mid u \leq r, v \leq r\}$$

and therefore

$$|P(r)| \leq \bigvee\{\delta(u,v) \mid u \leq r, v \leq r\}.$$

The proof of (22) is thus completed. □

PROPOSITION 8. *If* $\mathscr{R}$ *is a p-p-m-space (a p-m-space) satisfying* A6, $\delta = \delta_{|\,|}$ *and* $|\ | = |\ |_\delta$, *then*

(23) $\delta(r,s) = \delta(P(r),P(s))$ *and* $|r| = |P(r)|, \quad r,s \in R_0.$

As a consequence, if in R there is no empty region, the function $P: R \to R'$ *is a homomorphism (isomorphism) from* $\mathscr{R}$ *onto the canonical space* $\mathscr{R}'$.

PROOF. The first equality in (23) is an immediate consequence of (22). Moreover, since A4 holds for canonical distances and diameters,

$$\begin{aligned} |P(r)| &\geq \bigvee\{\delta(P(u),P(v)) - \delta(P(u),P(r)) - \delta(P(r),P(v)) \mid u,v \in R\} \\ &= \bigvee\{\delta(u,v) - \delta(u,r) - \delta(r,v) \mid u,v \in R\} = |r|_\delta = |r|. \end{aligned}$$

Then (23) follows from (20) of Proposition 6.

By (23) and (18), $P: R \to R'$ is a homomorphism. If $\mathscr{R}$ is a p-m-space and $P(x) \supseteq P(y)$, then from (23) it follows that $|x| \geq |y|$ and $\delta(z, x) = \delta(P(z), P(x)) \leq \delta(P(z), P(y)) = \delta(z, y)$. By A5, this proves that $x \geq y$. Thus $P: R \to R'$ is an isomorphism. □

Notice that also in the case that A1–A6 are satisfied, the map $P: R \to R'$ is not injective, in general, and it may happen that two different regions have the same points. For example the canonical space defined by the family $[-1/n, 1/n]$ of subsets of the real line satisfies A1–A6 but $P([-1/n, 1/n]) = \{0\}$. This leads to a search for other existential axioms. The following is proposed in [1].

A7 $$\forall y' \leq y\ \delta(y', x) = 0 \Rightarrow y \leq x.$$

In other words, A7 assures that if y is not contained in x, then a subregion y' of y exists such that $\delta(y', x) \neq 0$. A7 implies A5. Indeed, assume that $\delta(z, x) \leq \delta(y, z)$ for every $z \in R$. Then, for every $y' \leq y$ we have $\delta(y', x) \leq \delta(y, y') = 0$ and, by A7, $y \leq x$. From A7 it follows also that a minimum 0 exists in R if and only if $R = \{0\}$.

PROPOSITION 9. *If $\mathscr{R}$ satisfies* A1–A7, *then*

$$r \leq s \Leftrightarrow P(r) \subseteq P(s).$$

In other words, the map $P: R \to R'$ is an order-isomorphism.

PROOF. Assume that $P(r) \subseteq P(s)$ and that r is not contained in s. Then by A7 there exists $r' \leq r$ such that $\delta(r', s) \neq 0$. By A6 $P(r')$ and $P(s)$ are nonempty, and by (19) $\delta(P(r'), P(s)) \geq \delta(r', s) > 0$. Since $P(r') \subseteq P(r)$, this contradicts the hypothesis $P(r) \subseteq P(s)$. □

§6. Two further definitions of point. More restrictive definitions of point are proposed in [1] and [5]. Namely, in [1], we consider the set $S_1(\mathscr{R})$ of the decreasing sequences $r_1 \geq r_2 \geq \cdots$ of nonempty regions such that $\lim_n |r_n| = 0$, and only those elements p of $M(\mathscr{R})$ that can be represented by elements of $S_1(\mathscr{R})$ are called points. We denote the set of those points by $M_1(\mathscr{R})$ and we set $P_1(r) = P(r) \cap M_1(\mathscr{R})$. Notice that if A6 holds then $P_1(r)$ is nonempty for every nonempty region r. If $\mathscr{R}$ is the canonical p-m-space of the open balls of a Euclidean space E, then $M_1(\mathscr{R}) = M(\mathscr{R}) = E$. But, in general, $(M_1(\mathscr{R}), d)$ is a proper subspace of $(M(\mathscr{R}), d)$. For example, if $\mathscr{R}$ is the p-m-space with zero diameters associated to a metric space (M, d), then $(M_1(\mathscr{R}), d)$ coincides with (M, d) while $(M(\mathscr{R}), d)$ coincides with the completion of (M, d). If R is the set of intervals (a, b) in $\mathbf{R}$ with $a > 0$ or $b < 0$, then $M(\mathscr{R}) = \mathbf{R}$ while $M_1(\mathscr{R}) = \mathbf{R} - \{0\}$. These two examples show that $(M_1(\mathscr{R}), d)$ is not complete, in general, while, as proven in Proposition 6, $(M(\mathscr{R}), d)$ is complete.

PROPOSITION 10. *Assume* A6. *Then $(M(\mathscr{R}), d)$ is the completion of $(M_1(\mathscr{R}), d)$ and, for every $r \in R$, $P(r)$ is the closure of $P_1(r)$ in $(M(\mathscr{R}), d)$.*

PROOF. Let $p \in M(\mathscr{R})$ be represented by $\langle r_n \rangle$ and let $p_n \in M_1(\mathscr{R})$ be a point of r_n. Then, since

$$d(p, p_n) < d(p, r_n) + d(r_n, p_n) + |r_n| = d(p, r_n) + |r_n|,$$

we have $\lim d(p, p_n) = 0$. Thus, every element of $M(\mathscr{R})$ is a limit of a sequence of elements of $M_1(\mathscr{R})$; and therefore, by the completeness of $(M(\mathscr{R}), d)$, the space $(M(\mathscr{R}), d)$ is the completion of $(M_1(\mathscr{R}), d)$. If $p \in P(r)$, since the above-defined points

p_n are elements of $P_1(r)$, p belongs to the closure of $P_1(r)$. Since $P(r)$ is closed, this completes the proof. □

Another definition of point is proposed in [5] via the relation $\ll$ defined by

$$r' \ll r \Leftrightarrow r' \leq r \text{ and } (\exists \lambda > 0\, \forall e(\delta(r', e) + |e| < \lambda \Rightarrow e \leq r)). \tag{24}$$

In a sense $r' \ll r$ means that r' is contained in r but is not internally tangent to r. The relation $\ll$ is transitive, namely

$$r' \ll r'' \text{ and } r'' \leq r \Rightarrow r' \ll r; \qquad r' < r'' \text{ and } r'' \ll r \Rightarrow r' \ll r.$$

Denote by $S_2(\mathscr{R})$ the set of *strongly decreasing* chains of nonempty regions $r_1 \gg r_2 \gg \cdots$ with vanishing diameters, and by $M_2(\mathscr{R})$ the set of the elements of $M(\mathscr{R})$ that can be represented by elements of $S_2(\mathscr{R})$. If $\mathscr{R}$ is the p-m-space of the open balls of a Euclidean space E, then $M_1(\mathscr{R}) = M_2(\mathscr{R}) = M(\mathscr{R}) = E$. If $R = \{(a,b) \mid b \leq 0 \text{ or } a \geq 0\}$ then $M(\mathscr{R}) = M_1(\mathscr{R}) = \mathbf{R}$ while $M_2(\mathscr{R}) = \mathbf{R} - \{0\}$. If $\mathscr{R}$ is the zero diameter p-m-space associated to a metric space (M, d), then $M_2(\mathscr{R})$ is the set of the isolated points of (M, d). This shows that also in the case that A6 and A7 are satisfied, it is possible that $M_2(\mathscr{R})$ is empty.

Namely, in [5] one proceeds in a slightly different way by considering in $S_2(\mathscr{R})$ the equivalence relation $\equiv^*$ defined by setting $\langle p_n \rangle \equiv^* \langle q_n \rangle$ provided that

$$\forall i\, \exists j\; p_i \geq q_j \quad \text{and} \quad \forall j\, \exists i\; q_j \geq p_i. \tag{25}$$

The *points* are the equivalence classes $p = [\langle p_n \rangle]^*$ modulo $\equiv^*$ determined by the elements $\langle p_n \rangle$ of $S_2(\mathscr{R})$. The distance d^* is defined as usual via the equality (12). We denote by $(M^*(\mathscr{R}), d^*)$ the metric space defined in this way. The following proposition shows that this space coincides with $(M_2(\mathscr{R}), d)$.

PROPOSITION 11. *If $\langle p_n \rangle$ and $\langle q_n \rangle$ are elements of $S_2(\mathscr{R})$ and $\langle p_n \rangle \equiv^* \langle q_n \rangle$, then $\langle p_n \rangle \equiv \langle q_n \rangle$. Then the equality $h([\langle p_n \rangle]^*) = [\langle p_n \rangle]$ defines a map $h\colon M^*(\mathscr{R}) \to M_2(\mathscr{R})$, and this map is an isomorphism between $(M^*(\mathscr{R}), d^*)$ and $(M_2(\mathscr{R}), d)$.*

PROOF. We limit ourselves to proving that h is injective. Indeed, assume that $[\langle p_n \rangle] = [\langle q_n \rangle]$ with $\langle p_n \rangle, \langle q_n \rangle \in S_2(\mathscr{R})$. Then, since $\langle \delta(p_i, q_i) \rangle$ is increasing and $\lim \delta(p_i, q_i) = 0$, we have that $\delta(p_i, q_i) = 0$, for every $i \in \mathbf{N}$, and therefore $\delta(p_i, q_j) = 0$ for every $i, j \in \mathbf{N}$. Let i be any index and let $\lambda > 0$ be a real number for which $\delta(p_{i+1}, e) + |e| < \lambda \Rightarrow e \leq p_i$. From $\lim |q_n| = 0$ it follows that there is an integer j such that $|q_j| < \lambda$ and, since $\delta(p_{i+1}, q_j) + |q_j| = |q_j| < \lambda$, we have that $q_j \leq p_i$. Thus we have proven that $\forall i\, \exists j\; p_i \geq q_j$; in the same way one proves that $\forall i\, \exists j\; q_i \geq p_j$, and this entails that $\langle p_n \rangle \equiv^* \langle q_n \rangle$. The surjectivity of h is immediate. □

As a consequence of Proposition 11, the definition of the equivalence $\equiv^*$ proposed in [5] can be simplified. For example, we can require merely that for every $i \in \mathbf{N}$ there exists $j \in \mathbf{N}$ such that p_i overlaps q_j. Indeed, in this case, $\lim \delta(p_n, q_n) = 0$ and hence, by the injectivity of h, $\langle p_n \rangle \equiv^* \langle q_n \rangle$.

We set $p\, \varepsilon_2\, r$, for $p \in M_2(\mathscr{R})$ and $r \in R$, provided that p is represented by a strongly decreasing sequence $p_1 \gg p_2 \gg \cdots$ with $p_1 \leq r$. In this case, from (25) it follows that if $q_1 \gg q_2 \gg \cdots$ is another representative of p, then there is $i \in \mathbf{N}$ such that $q_i \leq p_1 \leq r$. This means that an equivalent way of defining ε_2 is to set $p\, \varepsilon_2\, r$ iff for every sequence $p_1 \gg p_2 \gg \cdots$ representative of p there is $i \in \mathbf{N}$ such that $p_i \leq r$. We set $P_2(r) = \{p \in M_2(\mathscr{R}) \mid p\, \varepsilon_2\, r\}$ for every $r \in R$.

Notice that ε_2 is different from the restriction of ε to $M_2(\mathscr{R})$ and that $P_2(r)$ is different from $P(r) \cap M_2(\mathscr{R})$, in general. As an example, let

$$R = \{(0, 1/n) \mid n \in \mathbf{N}\} \cup \{(-1/n, 1/n) \mid n \in \mathbf{N}\};$$

then $M(\mathscr{R}) = M_2(\mathscr{R}) = \{0\}$, but $P((0,1)) = \{0\}$ and $P_2((0,1)) = \varnothing$.

The following proposition examines the connection between P and P_2.

PROPOSITION 12. *If $r' \ll r$, then $P(r') \subseteq (P(r))^0$. Moreover,*

$$P_2(r) = \bigcup\{P(r') \cap M_2(\mathscr{R}) \mid r' \ll r\} \tag{26}$$

and this implies $P_2(r) \subseteq (P(r))^0 \cap M_2(\mathscr{R})$. If $\mathscr{R}$ satisfies A6 *and* A7, *then*

$$P_2(r) = (P(r))^0 \cap M_2(\mathscr{R}). \tag{27}$$

PROOF. Let $r' \ll r$ and let λ be as in (24). We will prove that, for every $c \in P(r')$, the open ball $B(c, \lambda/2)$ with center c and radius $\lambda/2$ is contained in $P(r)$. Now, assume that $\langle c_n \rangle$ represents c, with $c_n \leq r'$, and that $p = \langle p_n \rangle \in B(c, \lambda/2)$. Then there is $\nu \in \mathbf{N}$ such that $\delta(p_n, c_n) < \lambda/2$, $|p_n| < \lambda/2$, for every $n \geq \nu$. Thus $\delta(p_n, r') + |p_n| \leq \delta(p_n, c_n) + |p_n| \leq \lambda$ and, by (24), $p_n \leq r$ for every $n \geq \nu$. This proves that $p \in P(r)$.

To prove (26), let p be an element of $P_2(r)$ represented by $\langle p_n \rangle \in S_2(\mathscr{R})$ with $p_1 \leq r$. Let $r' = p_2$. Obviously $p \in P(r')$ and $r' \ll r$, and this proves that $P_2(r)$ is contained in $\bigcup\{P(r') \cap M_2(\mathscr{R}) \mid r' \ll r\}$. Conversely, if $p \in P(r') \cap P_2(\mathscr{R})$ with $r' \ll r$, then p is represented both by $\langle p'_n \rangle \in S$ with $p'_n \leq r'$ for every $n \in \mathbf{N}$, and by $\langle p_n \rangle \in S_2(\mathscr{R})$. Since $\lim \delta(p_n, p'_n) = 0$ and $\lim |p_n| = 0$, a natural number ν exists such that $\delta(p_n, p'_n) < \lambda/2$ and $|p_n| < \lambda/2$ for every $n \geq \nu$. Thus $\delta(p_n, r') + |p_n| \leq \delta(p_n, p'_n) + |p_n| < \lambda$, and therefore $p_n \leq r$ for every $n \geq \nu$. This proves that $p \in P_2(r)$.

To prove (27), assume $p \in (P(r))^0 \cap P_2(\mathscr{R})$. Then $\lambda > 0$ exists such that $B(p, \lambda) \subseteq P(r)$. If p is represented by $\langle p_n \rangle \in S_2(\mathscr{R})$, then $\nu > 0$ exists such that $|p_\nu| < \lambda$. Since from $q \in P(p_\nu)$ and (20) it follows that $d(p, q) \leq |P(p_\nu)| \leq |p_\nu| < \lambda$, we have that $P(p_\nu) \subseteq B(p, \lambda) \subseteq P(r)$. By Proposition 8 this implies $p_\nu \leq r$, and therefore p is an element of $P_2(r)$. □

PROPOSITION 13. *If $M_2(\mathscr{R})$ is nonempty, then every $P_2(r)$ is open and the family $\{P_2(r) \mid r \in R\}$ is a base for the topology of $(M_2(\mathscr{R}), d)$.*

PROOF. To prove that each $P_2(r)$ is open, let $p \in P_2(r)$; then p is represented by a strongly decreasing sequence $\langle p_n \rangle$ such that $p_1 \leq r$. Since $p_2 \ll p_1$, a suitable $\lambda > 0$ exists such that $\delta(p_2, e) + |e| < \lambda$ implies $e \leq p_1$.

We will prove that the ball of $M_2(\mathscr{R})$ with center p and radius $\lambda/2$ is contained in $P_2(r)$. Now, let q be an element of $M_2(\mathscr{R})$ such that $d(q, p) < \lambda/2$, and assume that q is represented by an element $\langle q_n \rangle$ of $S_2(\mathscr{R})$. Let m be a natural number such that $|q_m| < \lambda/2$ and $\delta(p_m, q_m) < \lambda/2$. Thus

$$\delta(p_2, q_m) + |q_m| \leq \delta(p_m, q_m) + |q_m| \leq \lambda,$$

and this proves $q_m \leq p_1 \leq r$ and therefore $q \in P_2(r)$.

To prove that $\{P_2(r) \mid r \in R\}$ is a base, we have to prove that for every open ball $B(c, \rho)$ of $(M_2(\mathscr{R}), d)$ and $p \in B(c, \rho)$, there is $r \in R$ such that $p \in P_2(r)$ and $P_2(r) \subseteq B(c, \rho)$. Now, let $\langle p_n \rangle$ be an element of $S_2(\mathscr{R})$ representative of p. Since $\rho - d(p, c) > 0$ and $\lim |p_n| = 0$, an index h exists such that $|p_h| < \rho - d(p, c)$. We have that

$p \in P_2(p_h)$; moreover, $P_2(p_h) \subseteq B(c, \rho)$. Indeed from $q \in P_2(p_h)$ it follows that

$$d(q,c) \le d(q,p) + d(p,c) \le |p_h| + d(p,c) \le \rho - d(p,c) + d(p,c) = \rho. \qquad \square$$

7. **Some examples.** In this section we give some examples of p-p-m-spaces. Let X and Y be two nonempty sets, and denote by $F(X, Y)$ the class of partial functions from X to Y. If $f \in F(X, Y)$ we denote by D_f the domain of f. Given a finitely additive measure $\mu: \mathscr{P}(X) \to [0, 1]$, we set, for every $f, g \in F(X, Y)$,

$$\delta(f,g) = \mu(\{x \in X \mid x \in D_f \cap D_g \text{ and } f(x) \neq g(x)\}), \qquad |f| = \mu(\{x \in X \mid x \notin D_f\}).$$

Moreover, we define an order relation $\le$ by

$$f \le g \Leftrightarrow D_g \subseteq D_f \text{ and } g(x) = f(x)\ \forall x \in D_g.$$

In a sense $\delta(f,g)$ is a measure of the contrast between f and g, $|f|$ a measure of the indeterminateness of f, and $f \le g$ means that f is an extension of g. Notice that two elements f and g of $F(X, Y)$ overlap if and only if they admit a common extension, if and only if there is no $x \in X$ such that $f(x) \neq g(x)$. Moreover, f is an atom if and only if it is totally defined, while $|f| = 0$ if and only if f is almost-everywhere defined. Finally, the empty function is the maximum of $F(X, Y)$.

PROPOSITION 14. *Let $\mathscr{C}$ be a nonempty class of partial functions. Then $(\mathscr{C}, \le, \delta, |\ |)$ is a p-p-m-space.*

PROOF. A1, A2 and A3 are obvious. To prove A4, observe that, for every $f, g, h, \in \mathscr{C}$, the set

$$\{x \in D_f \cap D_g \mid f(x) \neq g(x)\}$$

is contained in

$$\{x \in D_f \cap D_h \mid f(x) \neq h(x)\} \cup \{x \in D_h \cap D_g \mid h(x) \neq g(x)\} \cup \{x \in X \mid x \notin D_h\}. \quad \square$$

PROPOSITION 15. *Let μ be a measure on the denumerable set X such that $\mu(\{x\}) \neq 0$ for every $x \in X$, and assume that there are at least two elements in Y. If $\mathscr{C}$ is the class of the finite partial functions from X into Y, then the space $(\mathscr{C}, \le, \delta, |\ |)$ satisfies* A1–A7, *$|\ | = |\ |_\delta$ and $\delta = \delta_{|\ |}$. The same holds if $\mathscr{C} = F(X, Y)$.*

PROOF. Let $x_1, x_2, \ldots$ be an enumeration of the elements of X, and set $X_n = \{x_1, \ldots, x_n\}$. To prove A6, let $f \in \mathscr{C}$ and $\varepsilon > 0$. Since $\bigvee \mu(X_n) = \mu(X) = 1$ and D_f is finite, an integer m exists such that $\mu(X_m) \ge 1 - \varepsilon$ and $D_f \subseteq X_m$. Let h be any extension of f whose domain is X_m; then $h \le f$ and $|h| = \mu(-X_m) \le \varepsilon$.

To prove A7, assume $g \not\ge f$. Then, in the case $D_g \subseteq D_f$, an element $a \in D_g$ exists such that $g(a) \neq f(a)$, and therefore $\delta(f,g) \ge \mu(\{a\}) \neq 0$. If D_g is not contained in D_f, let $a \in D_g - D_f$ and $y \in Y - \{g(a)\}$. Then if $h = f \cup \{(a, y)\}$ we have $h \le f$ and $\delta(h,g) \ge \mu(\{a\}) \neq 0$.

To prove that $\delta = \delta_{|\ |}$, let $f, g \in \mathscr{C}$ and let y be any element of Y. We set

$$h^n = \{(x, f(x)) \mid x \in D_f \cap D_g \text{ and } f(x) = g(x)\} \cup \{(x, g(x)) \mid x \in D_g - D_f\}$$
$$\cup \{(x, f(x)) \mid x \in D_f - D_g\} \cup \{(x, y) \mid x \in X_n - D_f \cup D_g\}.$$

The partial function h^n belongs to $\mathscr{C}$ and overlaps both f and g. Consequently, h^n is a path from f to g and, since $\{x \in X \mid h^n \text{ is not defined in } x\}$ is contained in

$\{x \in D_f \cap D_g \mid f(x) \neq g(x)\} \cup (-X_n)$, we have

$$\delta_{|\,|}(f,g) \leq |h^n| \leq \mu(\{x \in D_f \cap D_g \mid f(x) \neq g(x)\}) + \mu(-X_n) = \delta(f,g) + \mu(-X_n).$$

Since $\lim \mu(-X_n) = 0$, we obtain $\delta_{|\,|}(f,g) \leq \delta(f,g)$, and this proves, by Proposition 3, that $\delta_{|\,|} = \delta$.

Likewise, if $f \in \mathscr{C}$, let y_1 and y_2 be two different elements of Y and set $h_i^n = \{(x, y_i) \mid x \in X_n - D_f\} \cup \{(x, f(x) \mid x \in D_f\}$. Then $h_i^n \in \mathscr{C}$, $h_i^n \leq f$ and

$$|f|_\delta \geq \bigvee \delta(h_1^n, h_2^n) = \bigvee \mu(X_n - D_f) = \mu\left(\bigcup (X_n - D_f)\right) = \mu(-D_f) = |f|.$$

By Proposition 3 this proves that $|f|_\delta = |f|$. The case $\mathscr{C} = F(X, Y)$ is obvious. □

In the case $Y = \{0, 1\}$ the elements of $F(X, Y)$ may be interpreted as *partially defined* subsets of X. Interesting classes of partially defined subsets are furnished by mathematical logic. Namely, let X be equal to the set of the sentences of a first order language and let T be a theory, i.e. a consistent set of sentences closed with respect to the logical deductions. Then we identify T with

$$f_T = \{(x, 1) \mid x \in T\} \cup \{(x, 0) \mid \neg x \in T\},$$

where $\neg x$ denotes the negation of the formula x. In [3] we prove that, if $\mu(\{x\}) \neq 0$ for every $x \in X$, then the class of the (partial functions associated to the) axiomatizable theories defines a p-m-space satisfying A1–A7. In this space:

- $\delta(T, T') = \mu(\{x \in X \mid x \in T \text{ and } \neg x \in T' \text{ or } \neg x \in T \text{ and } x \in T'\})$ represents a measure of the contrast between the theories T and T'.
- $|T| = \mu(\{x \in X \mid x \in T \text{ or } \neg x \in T\})$ is a measure of the degree of incompleteness of the theory T.
- The relation $\leq$ is the opposite of the inclusion relation between theories.
- The $\mathscr{R}$-points coincide with the axiomatizable complete theories.
- The points coincide with the complete theories.
- The points of a region are its completions.
- Two theories overlap if and only if they are consistent.

Pointless metric spaces of this type are proposed [3] in connection with Popper's verisimilitude problem. Namely, since the truth is represented by a point V, the verisimilitude of a theory T is defined as a suitable decreasing function of $\delta(T, V)$ and $|T|$. A precise definition of convergence to the truth for sequences of theories is also possible.

The second example of p-p-m-space is related to fuzzy set theory. Let (M, d) be a metric space. Then a *fuzzy subset* of M is any map $s: M \to [0, 1]$ (see [4]). Given two fuzzy subsets s and s', we set $s \leq s'$ provided that $s(x) \leq s'(x)$ for every $x \in M$. A fuzzy subset $s: M \to [0, 1]$ is *crisp* if $s(x) \in \{0, 1\}$ for every $x \in M$. We identify the crisp fuzzy subsets with the subsets of M via the characteristic functions. With respect to $\leq$, the class of the fuzzy subsets of M is a lattice extending the lattice of the subsets of M. We say that $s: M \to [0, 1]$ is *nonempty* if there is $x \in M$ such that $s(x) = 1$. If $\alpha \in [0, 1)$, the open *α-cut* $O(s, \alpha)$ of s is the set $\{x \in M \mid s(x) > \alpha\}$ [4]. Let R be the class of nonempty fuzzy subsets of M and set, for every $s, s' \in R$,

$$(28) \qquad \delta^*(s, s') = \int_0^1 \delta(O(s, \alpha), O(s', \alpha))\, d\alpha, \qquad |s|^* = \int_0^1 |O(s, \alpha)|\, d\alpha,$$

where δ and $|\ |$ are defined by (1) and we set $\int_0^1 |O(s,\alpha)|\,d\alpha = \infty$ whenever there is $\alpha \in (0,1)$ such that $|O(s,\alpha)| = \infty$.

PROPOSITION 16. *The structure $\mathscr{R} = (R, \leq, \delta^*, |\ |^*)$ determined by a class of nonempty fuzzy subsets is a p-p-m-space. The points of this space are elements of the metric space (M,d).*

PROOF. A1, A2 and A3 are immediate. Let s, s' and t be nonempty fuzzy subsets. Then by integrating both the sides of the inequality

$$\delta(O(s,\alpha), O(s',\alpha)) \leq \delta(O(s,\alpha), O(t,\alpha)) + \delta(O(t,\alpha), O(s',\alpha)) + |O(t,\alpha)|$$

we obtain A4. The rest of proposition is obvious. □

Recall that a *fuzzy number* is defined as a nonempty fuzzy subset of the real line whose cuts are intervals. These numbers generalize the interval numbers via a suitable extension of the arithmetical operations. Proposition 16 suggests that, when passing from real numbers to fuzzy (interval) numbers, it seems useful to pass from metric spaces to p-p-m-spaces.

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