CHAPTER 18

Pointless Geometries

Giangiacomo GERLA

Dipartimento di Matematica ed Applicazioni, Monte S. Angelo, Via Cintia, I-80126 Napoli, Italy

Contents

HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

Introduction

Since the times of Euclid, the concept of point has been assumed as the main primitive term for an axiomatic foundation of geometry. Is this choice a necessary one? The answer is 'no', and in this chapter I shall expose some attempts that have been made to build up geometry by assuming as primitive the notions of region or solid, and thereafter defining the points in a suitable way.

In pointless geometry, regions are considered as individuals, i.e. in the vocabulary of logic, first order objects, while points are represented by classes (or sequences), i.e. second order objects. Obviously, expressions like 'pointless geometry' or 'geometry without points', as used in the literature and in the title of this chapter, should be understood as contractions of 'geometry without the point as a primitive concept'. As a matter of fact, the authors of the papers that will be examined just aim at giving a good definition of 'point'.

The choice of considering the concept of 'region' as primitive makes me exclude the most famous pointless geometry: the Von Neumann Continuous Geometry, which is based on the concept of closed subspace. The same reason makes me discard category theory, although this theory can be viewed as a no-point approach of the whole field of mathematics.

In Section 1, I shall sketch some historical information about pointless geometry. In Section 2, I shall examine pointless topology, and in Section 3 connection structure. Section 4 is devoted to pointless metric spaces and Section 5 to the physical geometry of H.J. Schmidt.

1. Historical remarks

1.1. *The first attempts*

The literature on pointless geometry is not too large and each author usually ignores the previous attempts at the subject.

As soon as 1835, Lobachevsky gives an example of pointless geometry, assuming as primitive the notions of solid and contact between two solids. Several types of contact, superficial, linear and pointwise, define the surfaces, lines and points, respectively. Unfortunately, Lobachevsky's definitions are a little obscure and far from a rigorous treatment. No subsequent paper on pointless geometries quotes Lobachevsky's geometry.

Several years later, Whitehead [1919] and [1920] analyses how the objects (e.g., volumes) and the relations (e.g., inclusion of volumes) supplied by nature can be combined in order to obtain an abstract notion of point. In Nicod [1930] such philosophical analysis is assumed as a starting point for a new approach to pure geometry. Later on, Whitehead [1929], following an idea of De Laguna [1922], introduces the connectedness relation between regions to replace the inclusion relation. In Grzegorczyk [1960] and Clarke [1981] precise axiom system for the connection relation are proposed (see Section 3).

In 1927, Tarski, in the framework of Lesniewski's mereology, sketches a solution to the problem of a geometry of solids based on the notions of sphere and inclusion between spheres. The concentricity relation is defined and enables us to define the points as complete classes of concentric spheres.

1.2. *More recent proposals*

The works of Ehresmann [1957/58], Benabou [1957/58], Papert and Papert [1957/58] give rise to pointless topology, i.e. an abstract treatment of a class of lattices (the frames) extending the class whose members are the lattices of open subsets of the topological spaces. Such lattices are interesting in the frameworks of topology, computability and intuitionistic logic (see Section 2).

Schmidt [1979] gives a complete treatment of pointless geometry in which, in addition to regions and inclusion between regions, the translations and rotations are assumed as primitive concepts (see Section 5). Weihrauch and Schreiber [1981] proposes a suitable system of axioms for the *partial orders with weight and distance*. Although such structures are examined in the framework of abstract computability theory, they turn out to be a promising starting point for a metrical approach of pointless geometry (see Section 4).

Finally, one may notice that a large number of papers related to time logics can perhaps be viewed as a chapter on pointless one-dimensional geometry. The *periods of time* (the regions) are assumed as primitive together with the *inclusion relation* and the *temporal order*, while the *instants* (the points) are defined in a suitable way (see, e.g., Hamblin [1971]). I do not consider these works in this chapter.

2. Pointless topologies

2.1. *Frames*

The class $\mathcal{T}$ of open subsets of a topological space constitutes a complete lattice such that

$$x \wedge \left(\bigvee x_i\right) = \bigvee (x \wedge x_i)$$

holds for every $x \in \mathcal{T}$ and every family $(x_i)_{i \in I}$ of elements of $\mathcal{T}$. In literature the complete lattices satisfying such distributive laws are called *frames* or *locals* and extensively examined. In the following, if $\mathcal{R} = (R, \leqslant)$ is a frame, we call *regions* the elements of R and *inclusion* relation the relation $\leqslant$.

The frames may be organized into a category FR by suitably defining morphisms. The definition is given in order to obtain a category extending that of topological spaces. Now, if f is a continuous map from a topological space $(X, \mathcal{T})$ into another $(X', \mathcal{T}')$, it determines, via f^{-1}, a map from $\mathcal{T}'$ to $\mathcal{T}$ preserving infinite joins and finite meets. This suggests calling *frame-morphism* from a frame $\mathcal{R}$ into a frame $\mathcal{R}'$, any lattice morphism from $\mathcal{R}'$ into $\mathcal{R}$ which preserves infinite joins.

2.2. *Points of a frame*

In the famous Birkhoff–Stone representation theorem for distributive lattices (see Johnstone [1982]), the points are identified with the prime filters. This is unsatisfactory from a geometrical point of view, because the built-up spaces always are totally disconnected, compact spaces, so that the usual geometrical spaces are excluded. As a matter of fact, by identifying the points with the prime filters, we obtain too many points, and in pointless geometry, a good definition of 'point' leads to consider some other types of filters.

In the case of frames, we define a *point* as any *completely prime* filter, i.e. a filter P such that, for every family $(x_i)_{i\in I}$, $\bigvee x_i \in P$ implies $x_i \in P$ for a suitable x_i. The points of a frame $\mathcal{R}$ are just the elements of $\mathcal{R}$ in the category FR. We denote by $\mathbb{P}$ the set of points and say that a point *belongs* to a region r, briefly $P \in r$, provided $r \in P$.

The following proposition shows that it is possible to identify $\mathbb{P}$ with the set of the $\wedge$-irreducible elements of $\mathcal{R}$. Remember that an element u of a distributive lattice L is $\wedge$-*irreducible* provided that, for every $x, y \in L$, $u \geqslant x \wedge y$ implies that either $u \geqslant x$ or $u \geqslant y$. In the lattice of the open subsets of a Euclidean space E, u is $\wedge$-irreducible if and only if it is the complement of a point of E.

PROPOSITION 1. *A class P of elements of $\mathcal{R}$ is a point if and only if there is a $\wedge$-irreducible region $u_{\mathcal{P}}$ such that P is the complement of the ideal generated by $u_{\mathcal{P}}$, i.e.*

$$P = \{x \in R\colon\ x \nleqslant u_{\mathcal{P}}\}.$$

The map π: $R \to \mathfrak{P}(\mathbb{P})$ defined by putting

$$\pi(r) = \{P \in \mathbb{P}\colon\ P \in r\} = \{P \in \mathbb{P}\colon\ r \in P\}$$

preserves infinite joins and finite meets and $\pi(\mathcal{R})$ is a topology on $\mathbb{P}$. If there are enough points in $\mathcal{R}$, then π is a lattice isomorphism between $\mathcal{R}$ and the topology $\pi(\mathcal{R})$.

PROPOSITION 2. *For every frame $\mathcal{R}$, the following propositions are equivalent*:

(i) *$\mathcal{R}$ is spatial, i.e. it is isomorphic to the lattice structure of a suitable topology*;

(ii) *every element x of $\mathcal{R}$ is a meet of $\wedge$-irreducible elements.*

Proposition 1 shows that the choice of open sets as primitive terms is somewhat unsatisfactory from the point of view of pointless geometry. Indeed, in a sense, the points are present directly in a spatial lattice under the form of (complements of) $\wedge$-irreducible elements, while it should be desirable to define them by an *abstraction* process as suggested by Whitehead. This leads to consider as 'privileged model' of the concept of region some particular type of open sets; e.g., regular open sets (a subset x is called *regular open*, or *regular* if x is the interior of its own closure).

3. Connection structure

3.1. *Whitehead's axioms*

In the following we call *connection structure* any pair $(\mathcal{R}, C)$ where $\mathcal{R}$ is a set and C a binary relation on $\mathcal{R}$. The elements of $\mathcal{R}$ are called *regions*, C is a *connection relation* and if xCy, then we say that x is *connected* with y. For any $z \in \mathcal{R}$, the set of all x's such that zCx is denoted by $C(z)$. Several binary relations can be defined in a connection structure. Namely, by setting

$$x \leqslant y \Leftrightarrow C(x) \subseteq C(y)$$

we obtain a preorder relation $\leqslant$ that we call *inclusion*. The *overlapping relation* $\bigcirc$ is defined by setting

$$x \bigcirc y \Leftrightarrow z \text{ exists such that } z \leqslant x \text{ and } z \leqslant y.$$

Finally, the *nontangential inclusion* $\ll$ is defined by

$$x \ll y \Leftrightarrow C(x) \subseteq \bigcirc(y),$$

where, for any $z \in \mathcal{R}$, $\bigcirc(z)$ is defined as $C(z)$.

Connection structures were first considered in De Laguna [1922] and successively in Whitehead [1929] where a very large sequence of properties that the connection structures had to verify is exposed (in Chapter 2, Whitehead exposed 31 assumptions!). The aim was to analyze the abstraction process leading to the notions of point, line and surface. No attempt was made by Whitehead to frame his analysis into a mathematical theory. In particular, no attempt was made to reduce his system of assumptions and definitions to a logical minimum. Also, it is not clear whether such a system is able to define the Euclidean geometry or not. The following set of axioms is equivalent to the first 12 assumptions (see Gerla and Tortora [1992]).

(A1) C is symmetric.
(A2) There is no maximum for $\subseteq$.
(A3) For every x and y there exists z connected with both x and y.
(A4) C is reflexive.
(A5) $C(x) = C(y) \Rightarrow x = y$.
(A6) Any region z contains two regions x and y that are not connected.

The points are defined by the basic notion of *abstractive set*. An abstractive set is a set α of regions such that

– α is totally ordered by the nontangential inclusion,
– there is no region included in every element of α.

Intuitively, an abstractive set can 'converge' either to a point, a line or an area. We say that an abstractive set α *covers* an abstractive set β if every region in α contains a region in β. A corresponding equivalence relation is defined by setting $\alpha \equiv \beta$ provided that α covers β and β covers α. Any complete class of equivalence modulo $\equiv$ is called a *geometrical element*. The covering relation induces an order relation in the class of geometrical elements: any minimal geometrical element is called a *point*.

Whitehead, in order to define the concept of straight segment, assumed that a class of regions exists whose elements are called *ovals*. The idea was that the ovals are the convex regions of the Euclidean space. Obviously, suitable properties were assumed for the class of ovals. The *straight segment* between two points P and Q is defined as the minimal geometrical element defined by an abstractive set covering P and Q whose elements are ovals.

3.2. *Grzegorczyk's axioms*

Grzegorczyk [1960] added to the primitive $\leqslant$, the relation of *being separated*. In order to emphasize the similarity with Whitehead's ideas, I assume as primitive the negation of this relation, namely the relation C of *being connected*.

Then, Grzegorczyk's axioms become:

(G_0) $(\mathcal{R}, \leqslant)$ is a mereological field, where a *mereological field* is the structure obtained by deleting the zero element in a complete Boolean algebra.
(G_1) C is reflexive.
(G_2) C is symmetric.
(G_3) If $x \leqslant y$ then $C(x)$ is included in $C(y)$.

We say that a set p of regions is *representative* of a point if:

(i) p is without minimum and totally ordered with respect to $\ll$;
(ii) given any two regions u and v, $u \bigcirc x$ and $v \bigcirc x$ for every $x \in p$ implies uCv.

We denote by S the class of representatives of points and we call a *point* the filter $\mathcal{P}$ generated by an element p of S. Notice that two elements p and p' of S define the same point provided that, for every $x \in p$ there exists $y \in p'$ such that $x \geqslant y$, and for every $y \in p'$ there exists a $x \in p$ such that $y \geqslant x$.

Moreover, we say that a point $\mathcal{P}$ *belongs* (is *adherent*) to a region r provided that r is an element of $\mathcal{P}$ (r overlaps with every element of $\mathcal{P}$). We denote by $\mathbb{P}$ the set of points and by $\mathcal{P}(r)$ the set of points belonging to r. The following two axioms deal with the existence of points.

(G_4) Every region has at least one point.
(G_5) If xCy, then there is a point $\mathcal{P}$ such that $\mathcal{P}$ is adherent to both x and y.

Grzegorczyk proves the following two basic theorems.

THEOREM 1. *Let $\mathcal{T}$ be a Hausdorff topology, $\mathcal{R}$ the class of nonempty regular elements of $\mathcal{T}$ and put, for every $x, y \in \mathcal{R}$, xCy if $\bar{x} \cap \bar{y} \neq \varnothing$. Then $(\mathcal{R}, \subseteq, C)$ satisfies* (G_0)–(G_3). *Moreover, if every point is the intersection of a decreasing (with respect to $\ll$) family of open sets, then $(\mathcal{R}, \subseteq, C)$ also satisfies* (G_4)–(G_5).

THEOREM 2. *Assume that $(\mathcal{R}, \subseteq, C)$ satisfies* (G_0)–(G_5), *and let $\mathfrak{T}$ be the topology on $\mathbb{P}$ generated by $\{\pi(x)\colon x \in \mathcal{R}\}$, then:*

(i) *$\{\pi(x)\colon x \in \mathcal{R}\}$ is the class of the nonempty regular elements of $\mathfrak{T}$;*

(ii) $x \leqslant y \Leftrightarrow \pi(x) \subseteq \pi(y)$;

(iii) $x \ll y \Leftrightarrow \overline{\pi(x)} \subseteq \pi(y)$;

(iv) $xCy \Leftrightarrow \overline{\pi(x)} \cap \overline{\pi(y)} \neq \varnothing$;

(v) $\mathcal{P}$ *is adherent to* $x \Leftrightarrow \mathcal{P} \in \overline{\pi(x)}$.

Let $\mathcal{T}$ be a topology and $(\mathcal{R}, \leqslant, C)$ the connection structure associated with it by Theorem 1. It is an open question as to whether the topological space obtained in Theorem 2 coincides with $\mathcal{T}$ (at least for the most usual topological spaces).

3.3. *The system of B.L. Clarke*

A more direct reference to Whitehead [1929] can be found in Clarke [1981, 1985]. Clarke considers structures of type $(\mathcal{R}, C)$ for which the following axioms hold.

(A_1) C is reflexive.

(A_2) C is symmetric.

(A_3) If $C(x) = C(y)$ then $x = y$.

(A_4) If $X \subseteq \mathcal{R}$ and X is nonempty, then X admits fusion, where x is the *fusion* of X provided that $C(x) = \bigcup \langle C(z)\colon\ z \in X \rangle$.

A *point* is defined as a nonempty set $\mathcal{P}$ of regions such that:

(i) if $x \in \mathcal{P}$ and $y \in \mathcal{P}$ then xCy;

(ii) if $x \in \mathcal{P}$, $y \in \mathcal{P}$ and $x \bigcirc y$, then $x \wedge y \in \mathcal{P}$;

(iii) if $x \in \mathcal{P}$ and $y \geqslant x$ then $y \in \mathcal{P}$;

(iv) if $x \vee y \in \mathcal{P}$ then $x \in \mathcal{P}$ or $y \in \mathcal{P}$.

As usual, we say that a point $\mathcal{P}$ *belongs* to a region x, and write $\mathcal{P} \in x$, provided that $x \in \mathcal{P}$. Clarke suggests the following existence axioms:

(A_5) if xCy then there exists a point $\mathcal{P}$ such that $\mathcal{P} \in x$ and $\mathcal{P} \in y$

which is the reciprocal of (i).

The following proposition (see Biacino and Gerla [1991]) shows that, in a sense, the system (A_1)–(A_5) characterizes the mereological fields, i.e. the complete Boolean algebras.

THEOREM 3. *If* $(\mathcal{R}, C)$ *satisfies* (A_1)–(A_5), *then* $(\mathcal{R}, \leqslant)$ *is a mereological field, and* C *is the overlapping relation. Conversely, if* $(\mathcal{R}, \leqslant)$ *is a mereological field and* C *is the overlapping relation, then* $(\mathcal{R}, C)$ *satisfies* (A_1)–(A_5).

The fact that the connection relation coincides with the overlapping relation seems far from Whitehead's purposes, but it is in accordance with Leonard and Goodman [1940].

4. The metrical approach

4.1. *Pointless metrical spaces*

In the previous sections we were still at a topological level; to justify the word 'geometry' we have to consider richer structures. In this section this is achieved by considering metrical concepts. We call a *pointless pseudometric space*, briefly *ppm-space*, any structure $\mathcal{R} = (R, \leqslant, \delta, \|)$ where $(R, \leqslant)$ is a partial order and $\|: R \to [0, \infty]$, δ: $R \times R \to [0, \infty)$ are functions satisfying, for every $x, y, z \in R$, the following axioms.

(A_1) If $x \geqslant y$ then $|x| \geqslant |y|$.
(A_2) If $x \geqslant y$ then $\delta(y, z) \geqslant \delta(z, x)$.
(A_3) $\delta(x, x) = 0$.
(A_4) $\delta(x, y) \leqslant \delta(x, z) + \delta(z, y) + |z|$.

A similar set of axioms was first defined by Weihrauch and Schreiber [1981] in the framework of computability theory (see also Pultr [1984a,b,c, 1989]). We call the number $\delta(x, y)$ the *distance* between x and y, and $|x|$ the *diameter* of x, and we say that x is *bounded* if $|x|$ is finite. If there exists in $(R, \leqslant)$ a minimum region, say O, we call O the *empty region*. Notice that, as with the pseudometric spaces, every (nonempty) subset of a ppm-space defines a ppm-space. From (A_1)–(A_4) it follows that $\delta(x, y) = \delta(y, x)$ and that if x and y overlap, then $\delta(x, y) = 0$.

If R is equal to a class of nonempty subsets of a pseudometric space (M, d), taking $\leqslant$ as the inclusion relation and δ and $\|$ as the usual distance and diameter functions defined by

$$\delta(X, Y) = \inf\{d(x, y)\colon\ x \in X,\ y \in Y\}$$

and

$$|X| = \sup\{d(x, y)\colon\ x \in X,\ y \in X\},$$

we then obtain a ppm-space. We call *canonical* any space obtained by these means.

We call *pointless metric space*, briefly *pm-space*, any ppm-space satisfying

(A_5) if $|x| \geqslant |y|$ and $\delta(z, x) \leqslant \delta(y, z)$ for every $z \in R$, then $x \geqslant y$.

Notice that ppm-spaces (pm-spaces) generalize pseudometric (metric) spaces; the pseudometric (metric) spaces being the ppm-spaces (pm-spaces) in which every region is an atom whose diameter is zero.

4.2. *Definition of diameters and distances*

Let $(R, \leqslant)$ be an ordered set and $\|: R \to [0, \infty]$ be a function. For $x, y \in R$, a path from x to y is any finite sequence $b_1, \ldots, b_n$, of nonempty bounded regions such that b_{i+1} overlaps b_i for each $i = 1$ to $n - 1$, x overlaps b_1 and y overlaps b_n. We say that

$(R, \leqslant, \|)$ is *connected* if for every pair of nonempty elements of R there is a path from x to y. We define the function $\delta_{\|}$: $R \times R \to [0, \infty]$ by $\delta_{\|}(x, y) = 0$ if $x \bigcirc y$, and

$$\delta_{\|}(x, y) = \inf \{ |b_1| + \cdots + |b_n| \colon b_1 \ldots b_n \text{ is a path from } x \text{ to } y \}$$

otherwise. This definition was given in Weihrauch and Schreiber [1981]. Moreover, for every function δ: $R \times R \to [0, \infty)$ we put:

$$|z|_\delta = \sup \{ \delta(x, y) - \delta(x, z) - \delta(z, y) \colon x, y \in R \} .$$

PROPOSITION 1. *Let $(R, \leqslant)$ be an ordered set together with a function $\|$: $R \to [0, \infty)$ satisfying (A_1) and assume that $(R, \leqslant, \|)$ is connected. Then $(R, \leqslant, \delta_{\|}, \|)$ is a ppm-space. If δ: $R \times R \to [0, \infty)$ is a function satisfying (A_2) and (A_3), then $(R, \leqslant, \delta, \|_\delta)$ is a ppm-space.*

PROPOSITION 2. *If $(R, \leqslant, \delta, \|)$ is a ppm-space, then $(R, \leqslant, \delta, \|_\delta)$ is a ppm-space such that $\|_\delta \leqslant \|$, that is, $\|_\delta$ is the smallest diameter compatible with δ. If $(R, \leqslant, \|)$ is connected, then $(R, \leqslant, \delta_{\|}, \|)$ is a ppm-space such that $\delta_{\|} \geqslant \delta$, that is, $\delta_{\|}$ is the largest distance compatible with $\|$.*

Notice that if $\mathcal{R}$ is a ppm-space, then, in the general case, δ and $\delta_{\|}$ are not the same, neither are $\|$ and $\|_\delta$.

4.3. *The points*

In Weihrauch and Schreiber [1981], points are defined as in Section 3.2, except that $\ll$ is defined by stating that $x \ll y$ if there exists a positive λ such that $\delta(x, z) + |z| < \lambda \Rightarrow z \leqslant y$. In Gerla [1990], I define the points by a procedure similar to the completion of a metric space using Cauchy sequences. We call a *Cauchy sequence* every sequence $\langle p_n \rangle$ of bounded regions such that

(i) $\lim |p_n| = 0$ and (ii) $\forall \varepsilon > 0\ \exists \nu \colon\ \delta(p_h, p_k) < \varepsilon\ \ \forall h \geqslant \nu,\ \forall k \geqslant \nu$.

Decreasing sequences with vanishing diameters are examples of Cauchy sequences. There is no need to have any Cauchy sequence in a ppm-space.

PROPOSITION 3. *Assuming that the class S of Cauchy sequences of $\mathcal{R}$ is nonempty and defining d: $S \times S \to [0, \infty)$ by $d(\langle p_n \rangle, \langle q_n \rangle) = \lim \delta(p_n, q_n)$ for every $\langle p_n \rangle \in S$ and $\langle q_n \rangle \in S$, (S, d) is a pseudometric space.*

We denote by $(\mathbb{P}, d)$ the metric space obtained as a quotient of (S, d) by the relation $\equiv$ defined by $\langle p_n \rangle \equiv \langle q_n \rangle$ if $d(\langle p_n \rangle, \langle q_n \rangle) = 0$. Moreover, we call a point every element of $\mathbb{P}$. As a consequence, a point P is a class

$$[\langle p_n \rangle] = \{ \langle q_n \rangle \in S \colon \langle q_n \rangle \equiv \langle p_n \rangle \}$$

and d: $\mathbb{P} \times \mathbb{P} \to [0, \infty)$ is defined by putting, for every P, Q in $\mathbb{P}$:

$$d(P, Q) = d\big(\langle p_n \rangle, \langle q_n \rangle\big) = \lim \delta(p_n, q_n)$$

where $\langle p_n \rangle \in S$ and $\langle q_n \rangle \in S$ represent P and Q, respectively.

If the pm-space $\mathcal{R}$ is a metric space, then the associated metric space $(\mathbb{P}, d)$ obviously is its completion. If $\mathcal{R}$ is the canonical pm-space of open balls of a Euclidean space E, then $(\mathbb{P}, d)$ coincides with E.

If $P \in \mathbb{P}$ and $r \in \mathcal{R}$, we say that P *belongs* to r, briefly $P \in r$, provided that there is a sequence $\langle p_n \rangle$ representing P with $p_n \leqslant r$ for every $n \in N$. We denote by $\pi(r)$ the set of all points belonging to r.

Axioms (A_1)–(A_5) do not guarantee the existence of points in a ppm-space. In order to get this, we have to add some new axiom. For example, we may assume that every region contains arbitrarily small regions.

(A_6) $\forall \varepsilon > 0 \ \forall r \in \mathcal{R} \ \exists r' \leqslant r$ such that $|r'| < \varepsilon$.

Obviously, (A_6) is equivalent to saying that every region has points. In this case, the class $\mathcal{R}' = \pi(\mathcal{R}) = \{\pi(r) \colon r \in \mathcal{R}\}$ of subsets of $\mathbb{P}$ defines a canonical ppm-space $\mathcal{R}'$, the *canonical ppm-space associated with* $\mathcal{R}$.

THEOREM 4. *Assume that* $\mathcal{R}$ *is a ppm-space satisfying* (A_6). *Then*:

(i) $(\mathbb{P}, d)$ *is a complete metric space*;

(ii) *for every* $r \in R$, $\pi(r)$ *is a closed subset*;

(iii) *if* $r \leqslant s$, *then* $\pi(r) \subseteq \pi(s)$;

(iv) $\delta(r, s) \leqslant \delta(\pi(r), \pi(s)) \leqslant \delta_{||}(r, s)$ *and* $|\pi(r)| = \sup\{\delta(u, v) \colon u \leqslant r,\ v \leqslant r\}$.

Moreover, if $\delta = \delta_{||}$ *and* $|| = ||_\delta$, *then the function* π: $\mathcal{R} \to \mathcal{R}'$ *is an isomorphism between* $\mathcal{R}$ *and its associated canonical space* $\mathcal{R}'$.

We conclude this section by noting that, since many classical geometries may be defined in terms of axioms about metric spaces, pointless metric geometry leads to complete axiomatizations of these geometries without the primitive notion of 'point'.

5. Physical geometry

5.1. *The first axioms*

Schmidt [1979] proposed perhaps the most complete treatment of Euclidean pointless geometry. The term 'physical geometry' means that this geometry is understood as a theory of the physical space. As a consequence, axioms are thought as empirical laws governing the behaviour of rigid bodies, rather than as a mathematical device to produce the desired theorems. The goal is to construct a set $\mathbb{P}$ of points, a topology N^{top} on $\mathbb{P}$ and a group $\overline{T}$ of transformations of $\mathbb{P}$ such that $(\mathbb{P}, N^{\text{top}}, \overline{T})$ is isomorphic to the Euclidean space E equipped with the usual topology and the group generated by the translations and the rotations.

The following are the first two axioms of physical geometry:

(R_1) $(R, \leqslant)$ is a *weakly distributive lattice* (i.e. $r \wedge z = 0$ and $r' \wedge z = 0$ implies $(r \vee r') \wedge z = 0$) with an empty region 0, and $R \neq \{0\}$.
(R_2) T is a group of automorphisms of $(R, \leqslant)$.

Let r and r' be two regions. If r contains each overlapping displacement of r', i.e. if $\tau r' \leqslant r$ for every $\tau \in T$ such that $r' \wedge \tau r' \neq 0$, then we say that r' is a *kernel* of r.

(R_3) Every nonempty region has a kernel.

As usual, the group T determines an equivalence relation in R; we call *shapes* the related equivalence classes.

(R_4) Each region can be covered by finitely many regions of any given shape (with possible overlappings).

5.2. *Points and Cauchy filters*

To define the points, Schmidt uses a construction similar to the completion of a uniform space by the use of minimal Cauchy filters. The points are defined as suitable filters of $(R, \leqslant)$ as is usual in lattice theory. Now, it is very natural to require that a point may be represented by regions as small as we wish and therefore to only consider filters F such that, for every nonempty region r, there exists $s \in F$ such that $s \leqslant \tau r$ for a suitable $\tau \in T$. Since F is a filter, this is equivalent to require that $\tau r \in F$. We thus have the following definition:

DEFINITION 1. A *Cauchy filter* (briefly, C-filter) is any filter of R containing regions of every shape.

Now, the C-filters are not suitable candidates for a definition of 'point'. In fact, it is possible that, in a sense, two different C-filters F and F' be *infinitely close*, i.e. for every $r \neq 0$ there is $\tau \in T$ such that $\tau r \in F$ and $\tau r \in F'$. A similar situation arises when we complete a metric space: it is possible that two different Cauchy sequences represent the same point. This leads us to identify two infinitely close filters F and F' and to call 'point' any complete equivalence class of the relation 'is infinitely close to'. Equivalently, it is possible to identify the points with suitable filters representative of such classes; such filters are the minimal C-filters. Indeed, two minimal, infinitely close C-filters coincide, and one may prove that every C-filter contains an infinitely close minimal filter. Thus we get the following definition.

DEFINITION 2. A *point* P of $(R, \leqslant, T)$ is any C-filter, minimal in the class of the C-filters.

We denote by $\mathbb{P}$ the set of points, and put $\pi(r) = \{P \in \mathbb{P}\colon r \in P\}$.
The following proposition shows that $\pi\colon R \to P(\mathbb{P})$ is a lattice representation in a weak sense.

PROPOSITION 1. *For every* $r, s \in R$:

(i) *if* $r \leqslant s$ *then* $\pi(r) \leqslant \pi(s)$;

(ii) $\pi(r) = 0$ *if and only if* $r = 0$;

(iii) $\pi(r \wedge s) = \pi(r) \cap \pi(s)$;

(iv) $\pi(r \vee s) \supseteq \pi(r) \cup \pi(s)$.

Notice that, since a C-filter is not always prime, in general in (iv) the equality does not hold. In other words, it is possible for P to be a point of the union $r \vee s$ of two regions r and s while being neither a point of r nor a point of s.

5.3. *Topological and metric structures*

The next step is to define a suitable topology on $\mathbb{P}$. Schmidt proceeds as follows: remember that a *uniform structure*, or *uniformity*, on a set X is a filter U of $X \times X$ such that:

(U_1) every element of U contains the diagonal of $X \times X$;
(U_2) if $V \in U$ then $V^{-1} \in U$;
(U_3) for each $V \in U$ there exists a $W \in U$ such that $W \circ W \subseteq V$.

Each element V of U is then called an *entourage*. If $x, y \in X$ and $(x, y) \in V$, we say that x and y are V-close. A *fundamental system of entourages* for U is any set B of entourages such that every entourage contains an element of B. Every uniformity induces a topology, in which the filter of neighbourhoods of a point x is defined by the sets $V(x) = \{y \in x\colon (x, y) \in V\}$ with $V \in U$.

Schmidt defines an uniformity on $\mathbb{P}$ with a fundamental system of entourages

$$N = \left(N_r\right)_{r \in R},$$

where

$$N_r = \left\{(P, Q) \in \mathbb{P} \times \mathbb{P}\colon\ P, Q \in \tau r \text{ for a suitable } \tau \in T\right\};$$

N^{top} is defined as the topology induced by this uniformity. One may prove that the elements of $\pi(R)$ are open and relatively compact with respect to this topology.

On account of the very rich structure of a physical space, it is also possible to define a metric in $\mathbb{P}$ by utilizing chains of regions. A *chain of shape* $[v]$ is a sequence $s_1, \ldots, s_n$ of regions of the same shape $[v]$ such that each s_i overlaps s_{i+1}. If P is a point of s_1 and Q a point of s_n, we say that $s_1, \ldots, s_n$ is a *chain between* P *and* Q. Given a shape $[v]$, we denote by $\lambda(P, Q, v)$ the minimal length of the chains of shape $[v]$ between P and Q, if such exists. If we choose as unity of length a pair of points, say u and u', and set $\delta(P, Q, v) = \lambda(P, Q, v)/\lambda(u, u', v)$, then the *distance* $d(P, Q)$ is defined by

$$d(P, Q) = \lim_{r \to 0} \delta(P, Q, r),$$

where this equality means that for any $\varepsilon > 0$ there exists a $r \neq 0$ such that for all $v \leqslant r$ we have $|\delta(P,Q,v) - d(P,Q)| < \varepsilon$.

Obviously the above definition of distance is meaningful only if, given any shape $[v]$, the following two statements hold:

– for every pair of points p and q, there exists a chain of shape $[v]$ between p and q;

– for every pair of points p and q, the above mentioned limit exists.

This forces us to retain two additional axioms. The first one specifies that every transport can be made of arbitrarily small transport, and so enables us to prove that two points always are connected by a suitable chain:

(R_5) For every $\sigma \in T$ and every $s_1, \ldots, s_n \in R \setminus \{0\}$, there exist $\tau_1, \ldots, \tau_m \in T$ such that $s_i, \tau_1 s_i, \tau_2 s_i, \ldots, \tau_m s_i, \sigma s_i$ is a chain of shape $[s_i]$ for $i = 1$ to n.

The formulation of axiom (R_6) requires a previous definition of the group $\overline{T}$ of congruent mappings of $\mathbb{P}$. Now, every element τ of T induces a map

$$\widetilde{\tau}\colon \mathbb{P} \to \mathbb{P} \quad \text{defined by} \quad \widetilde{\tau}(P) = \{\tau x\colon x \in P\}.$$

The map $\sim$ is a homomorphism from T into the permutation group on $\mathbb{P}$. Unfortunately, the group $\widetilde{T} = \{\widetilde{\tau}\colon \tau \in T\}$ is not an adequate candidate to represent the whole congruence group of $\mathbb{P}$. Indeed, $\widetilde{T}$ operates only *almost transitively*, and not transitively, in general. This means that, for every point P, the set $\{\widetilde{\tau}P\colon \widetilde{\tau} \in \widetilde{T}\}$ is dense in $\mathbb{P}$ but different from $\mathbb{P}$, in general. Moreover, $\widetilde{T}$ may be incomplete. Let us define the topology t on T, whose open subbase is constituted by the sets

$$T(r) = \{\tau \in T\colon \tau r \wedge r \neq 0\}, \quad r \neq 0,$$

and their images by translations. T is a topological group with respect to t, and t determines a topology $\tilde{t}$ on $\widetilde{T}$ by the way of the homomorphism $\sim$. One may prove that $\tilde{t}$ coincides with the topology of compact convergence c; however, $\widetilde{T}$ is not complete with respect to this very natural topology.

Consequently, Schmidt builds up a completion $\overline{T}$ of $\widetilde{T}$. Let us define by c^s and c^d the left and right uniformities determined by c, and set $c^z = c^s \vee c^d$. Then the group $\overline{T}$ is defined as the c^z-completion of $\widetilde{T}$.

It is possible to give the following further axiom ensuring the existence of $\lim_{v \to 0} \delta(P,Q,v)$. Remember that, given two points P and Q, the *orbit* $J_P Q$ is the subset $\{\tau Q\colon \tau \in \overline{T},\ \tau P = P\}$.

(R_6) There exist two points $P, Q \in \mathbb{P}$ such that the orbit $J_P Q$ dissects the space $\mathbb{P}$, i.e. $\mathbb{P} \setminus J_P Q$ is not connected.

Axioms (R_1)–(R_6) enable us to prove that the pair $(\mathbb{P}, d)$ is a complete, locally compact metric space and that, if another unity of length is chosen, the corresponding metric coincides with d, up to a constant factor.

5.4. *The basic theorems of physical geometry*

By (R_1)–(R_6) we are able to prove the following basic theorem.

THEOREM 2. $(\mathbb{P}, N^{top})$ *is a connected, locally compact, uniform Hausdorff space. Moreover,* $\overline{T}$ *is a complete group of uniform homeomorphisms acting transitively on* $\mathbb{P}$ *and at least one of its orbits dissects the space* $(\mathbb{P}, N^{top})$.

In accordance with the results of Freudenthal [1955/56], such properties guarantee that $(\mathbb{P}, N^{top}, \overline{T})$ belongs to a very small class of possible geometries. Euclidean geometry could be obtained by adding two axioms asserting that the curvature vanishes and that the space has dimension 3.

We first have to define the dimension of a physical space. Given a nonempty region r (the radius), we say that a set V of regions is *r-approximately overlapping* if there is a nonempty region c (the centre) such that every element v of V overlaps with a suitable τr containing c. The dimension of R is the smallest number N such that each region can be covered by a finite number of arbitrarily small regions such that at most $N+1$ are r-approximately overlapping. The following is a precise definition.

DEFINITION 3. The number $\dim R$ is the smallest number N such that, given two nonempty regions s and v, a covering $v_1, \ldots, v_n$ of s and a nonempty region r exist such that:

(i) for every i, $v_i \leqslant \tau v$ for a suitable $\tau \in T$;
(ii) at most $N+1$ elements of $v_1, \ldots, v_n$ are r-approximately overlapping.

(R_7) $\dim R = 3$.

The vanishing of the curvature is guaranteed by the following.

(R_8) There exists a neighbourhood $U \in N^{top}$ such that if u, v, x, y, z are points of U then $d(x,u) = d(z,u)$, $d(x,v) = d(y,v)$, $d(x,z) = 2d(u,z)$ and $d(x,y) = 2d(v,y)$ imply $d(z,y) = 2d(u,v)$.

Axioms (R_1)–(R_8) enable us to prove the main theorem of Schmidt [1979].

THEOREM 3 (Schmidt [1979]). *The structure* $(\mathbb{P}, N^{top}, \overline{T})$ *is isomorphic to the Euclidean structure* $(E, \mathcal{T}, I_S)$, *where* E *is the Euclidean three-dimensional space,* $\mathcal{T}$ *its natural topology and* I_S *the group generated by the translations and rotations of* E.

References

Banaschewski, B., and A. Pultr
[1989] Cauchy points of metric locales, *Can. J. Math.* **5**, 830–854.

Benabou, J.
[1957/58] Treillis locaux et paratopologies, *Seminaire C. Ehresmann*, Faculté des Sciences de Paris.

Biacino, L., and G. Gerla
[1991] Connection structures, *Notre Dame J. Formal Logic* **32**, 242–247.

Birkhoff, G.
[1933] On the combination of subalgebras, *Proc. Cambridge Philos. Soc.* **29**, 441–464.

Birkhoff, G., and J. von Neumann
[1936] The logic of quantum mechanics, *Ann. Math.* **37**, 823–843.

Clarke, B.L.
[1981] A calculus of individuals based on "connection", *Notre Dame J. Formal Logic* **22**, 204–218.
[1985] Individuals and points, *Notre Dame J. Formal Logic* **26**, 61–75.

De Laguna, T.
[1922] Point, line and surface as set of solids, *J. Philosophy* **9**, 449–461.

Ehresmann, C.
[1957/58] Gattungen von lokalen strukturen, *Jahresber. Deutsch. Math.-Verein* **60**, 59–77.

Freudenthal, H.
[1955/56] Neuere Fassungen des Riemann–Helmholtz-lieschen Raumproblems, *Math. Z.* **63**, 374–405.

Gerla, G.
[1990] Pointless metric spaces, *J. Symbolic Logic* **55**, 207–219.
[1991] Pointless metric spaces and fuzzy spaces, in: *Applications of Category Theory to Fuzzy Subsets*, eds E.P. Klement, V. Höhle and S.E. Rodabaugh, Kluwer, Dordrecht, pp. 236–244.
[1992] Distances, diameters and verisimilitude of theories, *Arch. Math. Logic* **31**, 407–414.

Gerla, G., and R. Tortora
[1992] La relazione di connessione in A.N. Whitehead: Aspetti matematici, *Epistemologia* **15**, 341–354.

Gerla, G., and R. Volpe
[1985] Geometry without points, *Amer. Math. Monthly* **92**, 707–711.

Glibowski, E.
[1969] The application of mereology to grounding of elementary geometry, *Studia Logica* **24**, 109–127.

Grzegorczyk, A.
[1960] Axiomatizability of geometry without points, *Synthese* **12**, 228–235.

Halmos, P.R.
[1963] *Lectures on Boolean Algebras*, Van Nostrand, New York.

Hamblin, C.L.
[1969] Starting and stopping, *Monist* **53**, 410–425.
[1971] Instants and intervals, *Studium Generale* **24**, 127–134.

Huntington, E.V.
[1912] A set of postulates for abstract geometry expressed in terms of the simple relation of inclusion, in: *Proc. 5th Int. Congress of Mathematicians, Cambridge*, Vol. 2, pp. 466–470.
[1913] Postulates for abstract geometry, *Math. Ann.* **73**, 522–559.

Isbell, J.R.
[1972] Atomless parts of spaces, *Math. Scand.* **31**, 5–32.

Jaskowski, J.
[1948] Une modification des definitions fondamentales de la géometrie des corps de M.A. Tarski, *Ann. Soc. Math. Polon.* **21**, 298–301.

Johnstone, P.T.
[1982] *Stone Spaces*, Cambridge University Press, Cambridge.
[1983] The point of pointless topologies, *Bull. Amer. Math. Soc.* **8**, 41–53.

Leonard, H.S., and N. Goodman
[1940] The calculus of individuals and its uses, *J. Symbolic Logic* **5**, 45–55.

Lobačevskij, N.I.
[1835] Novye načala geometry s polnoj teoriej parallel'nyh (New principles of geometry with complete theory of parallels), in: *Polnoe sobranie socinenij*, Vol. 2, Moscow–Leningrad.

Mayr, D.
[1983] A constructive-axiomatic approach to physical space and spacetime geometries of constant curvature by the principle of reproducibility, in: *Space, Time and Mechanics*, eds D. Mayr and G. Sussmann, Reidel, Dordrecht, pp. 105–123.

Menger, K.
[1940] Topology without points, *Rice Inst. Pamphlets* **27**, 80–107.

Mortensen, C., and G. Nerlich
[1978] Physical topology, *J. Philos. Logic* **7**, 209–223.
Nicod, J.
[1930] *Foundations of Geometry and Induction*, Harcourt, Brace and Co., New York.
Papert, D., and S. Papert
[1957/58] Sur les treillis des ouverts et les paratopologies, *Seminaire C. Ehresmann*, Faculté des Sciences de Paris.
Papert, S.
[1959] Which distributive lattices are lattices of closed sets? *Proc. Cambridge Philos. Soc.* **55**, 172–176.
[1964] An abstract theory of topological spaces, *Proc. Cambridge Philos. Soc.* **60**, 197–203.
Pultr, A.
[1984a] Pointless uniformities I. Complete regularity, *Comm. Math. Univ. Carolinae* **25**, 91–104.
[1984b] Pointless uniformities II. (Dia)metrization, *Comm. Math. Univ. Carolinae* **25**, 105–120.
[1984c] Remarks on metrizable locales, *Proc. 12th Winter School, Suppl. ai Rend. del Circ. Mat. di Palermo*, Serie II – no **6**, 247–258.
[1989] Categories of diametric frames, *Math. Proc. Cambridge Phil. Soc.* **105**, 285–297.
Roper, P.
[1980] Intervals and tenses, *J. Philos. Logic* **9**, 451–469.
Schmidt, H.J.
[1979] *Axiomatic Characterization of Physical Geometry, Lecture Notes in Physics*, Springer, Berlin.
Śniatycki, A.
[1968] An axiomatics of non-Desarguean geometry based on the half-plane as the primitive notion, *Dissertationes Mathematicae* **59**, 1–42.
Sullivan, T.
[1971] Affine geometry having a solid as primitive, *Notre Dame J. Formal Logic* **12**, 1–61.
[1972] The name solid as primitive in Projective Geometry, *Notre Dame J. Formal Logic* **13**, 95–97.
[1973] Tarski's definition of point in Banach spaces, *J. Geom.* **3**(2), 179–189.
[1973] The geometry of solids in Hilbert spaces, *Notre Dame J. Formal Logic* **14**, 575–580.
Suppes, P.
[1972] Some open problems in the philosophy of space and time, *Synthese* **24**, 298–316.
Tarski, A.
[1956] Foundation of the geometry of solids, in: *Logic, Semantics, Mathematics*, (translated by J.H. Woodger), Oxford University Press, Oxford, pp. 24–30.
Thomason, S.K.
[1984] On constructing instants from events, *J. Philos. Logic* **13**, 85–86.
Vickers, S.
[1990] *Topology via Logic*, Cambridge University Press, Cambridge.
Von Neumann, J.
[1960] *Continuous Geometries*, Princeton University, Princeton, NJ.
Weihrauch, K.
[1979] Metric spaces defined by weighted algebraic cpo's, in: Fundamentals of Computation Theory, Vol. 2, ed. L. Budach, Akademie-Verlag, Berlin, pp. 516–522.
Weihrauch, K., and U. Schreiber
[1981] Embedding metric spaces into CPO's, *Theor. Computer Sci.* **16**, 5–24.
Whitehead, A.N.
[1919] *An Inquiry concerning the Principle of Natural Knowledge*, Cambridge University Press, Cambridge.
[1920] *The Concept of Nature*, Cambridge University Press, Cambridge.
[1929] *Process and Reality*, Macmillan, New York.