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Multivalued Logic to Transform Potential into Actual Objects

Abstract. We define the notion of "potential existence" by starting from the fact that in multi-valued logic the existential quantifier is interpreted by the least upper bound operator. Besides, we try to define in a general way how to pass from potential into actual existence.

Keywords: Potential existence, point-free geometry, infinity, multi-valued logic.

1. Introduction

Let $\alpha(x)$ denote a vague property such as *"big"*, *"high"*, *"old"* in a domain D and assume that the extension of $\alpha(x)$ is modelized by a fuzzy subset $r : D \to [0,1]$ of D. Then, since in multi-valued logic the existential quantifier is interpreted by the least upper bound operator, it is possible that $\exists x\alpha(x)$ is true at degree 1 and at the same time that no element d in D exists which satisfies $\alpha(x)$ at degree 1. This occurs when a sequence $d_1, d_2, \ldots$ of elements in D exists such that $Sup\{r(d_n) : n \in N\} = 1$ in spite of the fact that $r(d) \neq 1$ for any $d \in D$. In other words, in first order multi-valued logic it is possible that an existential formula $\exists x\alpha(x)$ is true but that there is no *witness* of the existence claimed by this formula. Obviously, we can consider this a pathological behavior of first order multi-valued logic since the meaning we have to assign to such a kind of existence is not clear. Instead, in this note we try to exploit such a pathology in order to define a notion of *"potential existence"*. In fact, in the case $\exists x\alpha(x)$ is true but no object exists satisfying $\alpha(x)$ we can claim that *potentially* there is an element satisfying $\alpha(x)$ but that there is no *actual* element satisfying $\alpha(x)$. This interpretation suggests the following question:

Is it possible to transform a fuzzy model I into an "equivalent" fuzzy model I^* in such a way that any formula like $\exists x\alpha(x)$ is true in I^* if and only if $\alpha(x)$ is true for a suitable element?

Presented by **Melvin Fitting**; *Received* June 8, 2005

Studia Logica (2007) 86: 69–87
DOI: 10.1007/s11225-007-9046-9

In other words

Is it possible to transform a fuzzy model I into an "equivalent" fuzzy model I^* in which any potential existence is an actual existence?

A positive answer to this question should give a method to modelize the passage from potential existence into actual existence. In some previous papers we explore this idea in the particular frameworks of pointless geometry (see [8], [9] and [11]) and set theory (see [7]).

Note that this is only an exploratory paper and that all the results are either a matter of routine or early established (as an example, see in [4]). Indeed, its aim is simply to put some questions and to suggest some future researches. Also, I am aware that the expressions "actual " and "potential" have a long tradition in philosophy and, in general, they have a meaning which is rather far from the one considered in this paper. As an example in the paper the potentiality is not intended as the capacity that a thing has to be in a different and more complete state. As a matter of fact in this paper the expressions "actual existence" and "potential existence" are only a suggestive way to denote mathematical properties of multi-valued logic.

2. Preliminaries

In this section we recall the main definitions in multi-valued first order logic. While we refer constantly to the book of Chang and Keisler, we will utilize the more expressive terminology of fuzzy logic. A *first order multi-valued logic* L is defined by a set $h_1, ..., h_t$ of logical connectives with the related arity, and by the quantifier $\exists$. We assume that each h_i is interpreted by an operator $\underline{h}_i : [0,1]^n \to [0,1]$ and that, as usual, $\exists$ is interpreted by the *Sup* operator in the power set $P([0,1])$. We say that a multi-valued logic is *continuous* provided that all the functions $\underline{h}_i$ are continuous. A *first order language* $\boldsymbol{L}$ is defined by a set C of constants, a set *Fun* of function names and a set *Rel* of relation names with the related arity function $ar : Fun \cup Rel \to N$. The set F of formulas is defined as usual and we assume that in F there are also the constants $\mathbf{0}$ and $\mathbf{1}$ to denote *false* and *true*, respectively. A *multi-valued interpretation* or *fuzzy model* of $\boldsymbol{L}$ is a pair (D, I) where D is a nonempty set, we call the *domain*, and I is a map that associates:

- any constant $c \in C$ with an element $I(c) \in D$;

- any function name $g \in Fun$ such that $ar(g) = n$ with an $n-$ary function $I(g) : D^n \to D$;

- any relation name R such that $ar(R) = n$ with an n-ary fuzzy relation $I(R) : D^n \to [0,1]$.

The interpretation of a term t whose variables are among $x_1, ..., x_n$ as a function $I(t) : D^n \to D$ is defined as in classical logic. Let $\alpha(x_1, ..., x_n)$ be a formula whose variables are among $x_1, ..., x_n$ and let $d_1, ..., d_n$ be elements in D. Then we define the value $I(\alpha, d_1, ..., d_n)$ by recursion by setting:

$$I(\mathbf{0}, d_1, ..., d_n) = 0$$
$$I(\mathbf{1}, d_1, ..., d_n) = 1$$
$$I(R(t_1, ..., t_m), d_1, ..., d_n) = I(R)(I(t_1)(d_1, ..., d_n), ..., I(t_m)(d_1, ..., d_n))$$
$$I(h(\alpha_1, ..., \alpha_t), d_1, ..., d_n) = \underline{h}(I(\alpha_1, d_1, ..., d_n), ..., I(\alpha_t, d_1, ..., d_n))$$
$$I(\exists x_j \alpha, d_1, ..., d_n) = Sup\{I(\alpha), d_1, ..., d_{j-1}, d, d_{j+1}, ..., d_n) : d \in D\}.$$

If x_i is not free in $\alpha(x_1, ..., x_n)$, then $I(\alpha, d_1, ..., d_n)$ does not depend on d_i. Accordingly, if α is a sentence, we write $I(\alpha)$ instead of $I(\alpha, d_1, ..., d_n)$. Given two interpretations (D, I) and (D', I'), a *homomorphism* is a map $f : D \to D'$ such that

- $f(I(c)) = I'(c)$
- $f(I(g)(d_1, \ldots, d_n)) = I'(g)(f(d_1), \ldots, f(d_n))$
- $I(R)(d_1, ..., d_n) \leq I'(R)(f(d_1), ..., f(d_n))$.

A homomorphism f is a *full* provided that

- $I(R)(d_1, ..., d_n) = I'(R)(f(d_1), ..., f(d_n))$.

An *isomorphism* is a one-one homomorphism whose inverse is a homomorphism or, equivalently, a full one-one homomorphism. We say that (D', I') is an *elementary extension* of (D, I), if an injective map $f : D \to D'$ exists such that, for any formula α and $d_1, ..., d_n$ in D

$$I(\alpha, d_1, ..., d_n) = I'(\alpha, f(d_1), ..., f(d_n)).$$

A *congruence* in (D, I) is an equivalence relation $\equiv$ in D such that

$$d_1 \equiv d'_1, ..., d_n \equiv d'_n \Rightarrow I(R)(d_1, ..., d_n) = I(R)(d'_1, ..., d'_n).$$
$$d_1 \equiv d'_1, ..., d_n \equiv d'_n \Rightarrow I(g)(d_1, ..., d_n) \equiv I(g)(d'_1, ..., d'_n).$$

for any relation symbol R and any operation symbol g. The *quotient of* (D, I) *modulo* $\equiv$ is the fuzzy model (D^*, I^*) defined by setting

$$D^* = D/\equiv$$
$$I^*(c) = [I(c)]$$
$$I^*(g)([d_1], ..., [d_n]) = [I(g)(d_1, ..., d_n)]$$
$$I^*(R)([d_1], ..., [d_n]) = I(R)(d_1, ..., d_n),$$

where, for every $x \in D$, $[x] = \{x' \in D : x' \equiv x\}$ and $D/\equiv = \{[x] : x \in D\}$.

In this paper we refer only to first order Lukasievicz logic whose connectives, $\wedge$, &, $\rightarrow$, $\neg$ are interpreted by the minimum and the functions $x \odot y = Max\{x + y - 1, 0\}$, $x \rightarrow y = Sup\{z \in [0, 1] : x \odot z \leq y\}$ and $1 - x$, respectively.

3. Actual and potential existence: the example of wide set theory

The following definitions play an important role in this paper.

DEFINITION 3.1. Let $\alpha(x_1, ..., x_n)$ be a formula whose only free variable is x_j and let (D, I) be a fuzzy model. Then we say that the existential sentence $\exists x_j \alpha(x_j)$ is *actually true* if an element $d \in D$ exists such that $I(\alpha, d_1, ..., d_{j-1}, d, d_{j+1}, ..., d_n) = 1$. We say that $\exists x_j \alpha(x_j)$ is *only potentially true* if it is not actually true and $I(\exists x_j \alpha(x_j)) = 1$. We call *ground* a fuzzy model (D, I) in which there is no formula which is only potentially true.

We say also that there is an "actual existence" ("potential existence") of an object in the case where an existential formula is actually true (potentially true, respectively). Trivially any fuzzy model in which I assumes its values in a finite subalgebra of [0,1] is ground. In particular, all crisp models are ground. Besides, any fuzzy model whose domain is finite is ground.

In order to give an example of a multi-valued model which is not ground, let denote the usual first order language for set theory by $\boldsymbol{L}$. Then $\boldsymbol{L}$ contains the relation symbols $=$, $\in$, $\subseteq$, the constant $\emptyset$ and the symbols $\{.\}$, $\cup$, to denote the *singleton* and the *union* operations. Let $\boldsymbol{L}_e$ be the language obtained by adding to $\boldsymbol{L}$ the relation name EQ to denote a *graded equipotence relation.* The intended interpretation is that the objects we speak about are finite sets and that $I(EQ)(x, y)$ is the degree of equipotence between x and y. Besides, if we denote the formula $EQ(x, y) \leftrightarrow \mathbf{1}$ by $x \equiv y$, we interpret $\equiv$ as the usual equipotence relation. Finally, in accordance with the fact that a set x is infinite if and only if it is equipotent with $x \cup \{x\}$, we denote by $INF(x)$ the formula $EQ(x, x \cup \{x\})$. We consider a theory T in such a language which contains suitable formulas expressing the main properties of the class of finite sets and the formulas:

A1 $EQ(x, x)$ (*reflexivity*)

A2 $EQ(x, y) \rightarrow EQ(y, x)$ (*symmetry*)

A3 $EQ(x, z) \wedge EQ(z, y) \rightarrow EQ(x, y)$ (*transitivity*)

A4 $INF(x) \wedge (x \subseteq y) \rightarrow INF(y)$ (*monotonicity*)

A5 $\neg INF(\emptyset)$ ($\emptyset$ *is not infinite*)

A6 $(x \subseteq y \wedge \neg(x = y)) \rightarrow \neg(x \equiv y)$ (*no set is equipotent to a part*)

A7 $\exists x(INF(x))$ (*infinity axiom*)

Moreover, since we are interested in models in which the relation $\neg$ and $\subseteq$ are crisp, we assume also that T contains the axioms $\neg(x \in y \wedge \neg x \in y)$ and $\neg(x \subseteq y \wedge \neg x \subseteq y)$. Observe that we can interpret indifferently $INF(x)$ as the vague predicate *"to be wide"* or the predicate *"to be infinite"* (which we also consider as vague). In accordance, we call *wide set theory* the proposed theory T. The following proposition was proved in [7].

PROPOSITION 3.2. *A multi-valued model for wide set theory exists whose elements are the finite sets and in which A7 is only potentially satisfied.*

PROOF. Denote the class of all finite sets of any model of set theory by *Fin* and, given a finite set x, denote the cardinality of x by $cr(x)$. Furthermore, in *Fin* define the fuzzy relation $I(EQ)$ by setting

$$I(EQ)(x, y) = 1 \quad \text{if } x = y = \emptyset$$

$$I(EQ)(x, y) = \frac{Min\{cr(x), cr(y)\}}{Max\{cr(x), cr(y)\}} \quad \text{otherwise.}$$

Then, $I(INF)(x) = cr(x)/(cr(x) + 1)$. It is immediate that such a model satisfies ***A1***, ***A2***, ***A4, A5, A6***. To prove ***A3*** it is sufficient to prove that

$$I(EQ)(x, z) + I(EQ)(z, y) - 1 \leq I(EQ)(x, y).$$

By observing that it is not restrictive to assume that x, y and z are pairwise different and that $cr(x) \leq cr(y)$, we consider three cases:

Case 1. $cr(z) \leq cr(x) \leq cr(y)$. Then

$$I(EQ)(x, z) + I(EQ)(z, y) - 1 = cr(z)/cr(x) + cr(z)/cr(y) - 1$$
$$\leq cr(z)/cr(y) \leq cr(x)/cr(y) = I(EQ)(x, y).$$

Case 2.$(cr(x) \leq cr(z) \leq cr(y)$. In such a case,

$$I(EQ)(x, z) + I(EQ)(z, y) - 1 = cr(x)/cr(z) + cr(z)/cr(y) - 1$$
$$\leq [cr(x)/cr(z)] \cdot [cr(z)/cr(y)] = cr(x)/cr(y) = I(EQ)(x, y)$$

Case 3. $cr(x) \leq cr(y) \leq cr(z)$. In such a case

$$I(EQ)(x, z) + I(EQ)(z, y) - 1 = cr(x)/cr(z) + cr(y)/cr(z) - 1$$
$$\leq cr(x)/cr(z) \leq cr(x)/cr(y) = I(EQ)(x, y).$$

To prove ***A7***, it is sufficient to observe that $Sup_{n \in N}\{\frac{n}{n+1} : n \in N\} = 1$. It is evident that no finite sets satisfy *INF*. ■

We say that the fuzzy model defined in such a proof is a *canonical model of wide set theory.*

4. Another example: point-free geometry

The possibility of considering a geometry in which the notion of point is not assumed as a primitive was extensively examined by A. N. Whitehead in *An Inquiry Concerning the Principles of Natural Knowledge* and in *The Concept of Nature*. In these books the primitives are the *regions* and the *inclusion relation* between regions. As a matter of fact, as observed by Casati and Varzi in [3], these books can be a basis for a *"mereology"* i.e. an investigation into the set theoretical *part-whole* relation, rather than for a point-free geometry. Indeed, the inclusion relation is not geometrical in nature. So, it is not surprising the fact that later, in *Process and Reality*, Whitehead proposed a different approach in which the primitives are the *regions* and the *connection relation* while the inclusion relation is defined. In [11] one examine the possibility of restating Whitehead primitive approach by considering a *"graded inclusion"* instead of a *"crisp inclusion"* relation.

In this section we will reconsider this approach from the point of view of a multi-valued logic. To do this, it is useful to individuate a mathematical model of our intuitive idea of region in the space. This idea includes the balls and any continuous deformations of a ball and it excludes points, lines, surfaces and other "immaterial" things. This is in accordance with Whitehead's analysis which emphasizes that abstract objects of such a kind are the result of an *"abstraction process"*. A reasonable choice is to refer to the regular subsets of a metric space (M, δ). Indeed, denote by $cl : P(M) \to P(M)$ and $int : P(M) \to P(M)$ the closure and the interior operator, respectively. Then we define $reg : P(M) \to P(M)$ by setting $reg(x) = cl(int(x))$ and we call *regular closed set*, in brief *regular set*, any fixed point of *reg*. The class *Reg* of regular subsets of M is a complete Boolean algebra. In the case (M,δ) is the three dimensional Euclidean space, the points, the lines and the surfaces are not regular set. Instead, any continuous deformation of a closed ball is a regular set.

Consider a first order language with a relation name *IN* and denote by $x \leq y$ the formula $IN(x, y) \Leftrightarrow \mathbf{1}$. The intended meaning is that:

- the variables denotes *regions* in a geometrical space,
- $\leq$ is the *inclusion* relation
- *IN* denotes a *graded inclusion relation.*

Denote the formula $\forall y(y \leq x \Rightarrow IN(x,y))$ by $PNT(x)$. We interpret $PNT(x)$ indifferently either as the vague predicate *"to be tiny"* or as the predicate *"to be a point"* (we regard as vague, too). Then we consider a theory T in such a language containing the following axioms

A1 $IN(x,x)$ (reflexivity).

A2 $IN(x,y) \wedge IN(y,z) \Rightarrow IN(x,z)$ (transitivity).

A3 $(x \leq y) \wedge (y \leq x) \Rightarrow x = y$ (anti-simmetry).

A4 $\exists x(PNT(x))$ (point existence).

Also, we assume that T contains formulas expressing the main properties of the class of the regular bounded nonempty subsets of a metric space (where $\leq$ is interpreted by the set theoretical inclusion). We call such a theory *graded inclusion spaces theory.* Let (Re, I) be a fuzzy interpretation and let *incl* be the interpretation $I(IN)$ of the vague predicate *IN*. Then, (Re, I) satisfies ***A1***, ***A2***, ***A3*** if and only if

$a1)$ $incl(x,x) = 1$

$a2)$ $incl(x,y) * incl(y,z) \leq incl(x,z)$

$a3)$ $\leq$ *is an order relation*

Moreover, if $p : Re \to [0,1]$ is the extension of the formula *PNT*, then

$p(x) = Inf_{y \leq x} incl(x,y)$

and ***A4*** is satisfied if and only if $Sup_x Inf_{y \leq x} incl(x,y) = 1$, i.e.

$a4)$ *for every* $n > 0$ *a region* d_n *exists s.t.* $incl(d_n, y) \geq 1\text{-}1/n$ *for every* $y \leq d_n$.

Instead, $I(PNT(x), d) = 1$ if and only if d is an atom of $(Re, \leq)$. Indeed, $Inf_{y \leq x} incl(d,y) = 1$ if and only if, for every region $y \leq d$, $incl(d,y) = 1$ and therefore $y = d$. To show the existence of a graded inclusion space, given any metric space (M, δ) we set, for any $P \in M$ and x, y subsets of M,

$$\delta(P,x) = Inf\{\delta(P,Q) : Q \in x\}. \tag{4.1}$$

$$D(x) = Sup\{\delta(P,P') : P \in x, P' \in x\}. \tag{4.2}$$

$$e_\delta(x,y) = Sup\{\delta(P,y) : P \in x\}. \tag{4.3}$$

THEOREM 4.1. *A fuzzy model for graded inclusion spaces theory exists in which the domain is the class of nonempty regular bounded subsets of a metric space* (M, δ). *In such a model* $\leq$ *is the set-theoretical inclusion and*

$$p(x) = (1 - D(x)) \vee 0. \tag{4.4}$$

Moreover, if (M, δ) is an Euclidean space, A4 is only potentially satisfied.

PROOF. Denote the class of all nonempty, regular, bounded subsets by *Reg*. Then, we interpret *Incl* by the fuzzy relation $incl : Reg \times Reg \to [0, 1]$ defined by setting

$$incl(x, y) = 1 - (e_\delta(x, y) \wedge 1) \tag{4.5}$$

Then it is immediate that $a1$) is satisfied. To prove $a2$), note firstly that

$$e_\delta(x, z) \leq e_\delta(x, y) + e_\delta(y, z).$$

Indeed, given $P \in x$ and $Q \in z$,

$$\delta(P, z) \leq \delta(P, Q) + \delta(Q, y) \leq \delta(P, Q) + e_\delta(y, z)$$

and therefore

$$\delta(P, z) \leq Inf_{Q \in z}\delta(P, Q) + e_\delta(y, z) = \delta(P, y) + e_\delta(y, z).$$

So,

$$\begin{aligned} &e_\delta(x, z) = Sup\{\delta(P, z) : P \in x\} \leq Sup\{\delta(P, y) + e_\delta(y, z) : P \in x\} \\ &= Sup\{\delta(P, y) : P \in x\} + e_\delta(y, z) \\ &= e_\delta(x, y) + e_\delta(y, z). \end{aligned}$$

Now in the case that $incl(x, y) = 0$ or $incl(y, z) = 0$, $a2$) is obvious. Otherwise,

$$\begin{gathered} incl(x, z) \geq 1 - e_\delta(x, z) \geq 1 - [e_\delta(x, y))) + e_\delta(y, z)] = \\ 1 - e_\delta(x, y) + 1 - e_\delta(y, z)) - 1 = incl(x, y) * incl(y, z). \end{gathered}$$

To prove $a3$), note that, since y is a closed set, $x \subseteq y \Leftrightarrow e_\delta(x, y) = 0$ $\Leftrightarrow Incl(x, y) = 1$. This proves both $a3$) and that the associated order is the inclusion relation. To prove (4.4) we observe at first that

$$D(x) = Sup\{e_\delta(x, x') : x' \subseteq x\}. \tag{4.6}$$

Indeed, since x is regular, $D(x) = D(int(x))$. Let P and P' be elements in $int(x)$ and let $n \in N$ so that $B_n(P') \subseteq x$. Then, by observing that e_δ is order-reversing with respect to the second variable, we have

$$Sup\{e_\delta(x, x') : x' \leq x\} \geq e_\delta(x, B_n(P')) \geq \delta(P, B_n(P')) = \delta(P, P') - 1/n.$$

This entails that

$$\{e_\delta(x, x') : x' \subseteq x\} \geq lim_{n\to\infty}\delta(P, P') - 1/n = \delta(P, P')$$

and therefore that $Sup\{e_\delta(x, x') : x' \subseteq x\} \geq D(x)$. Conversely, since $e_\delta(x,x') \leq e_\delta(x,\{P'\})$ for any $P' \in x'$,

$Sup\{e_\delta(x,x') : x' \subseteq x\}$
$\leq Sup\{e_\delta(x,\{P'\}) : P' \in x\} = Sup_{P\in x}Sup_{P'\in x}e_\delta(\{P\},\{P'\}) = D(x),$

Taking into account (4.6), we have that

$p(x) = Inf\{incl(x,x') : x' \leq x\} = Inf\{1\text{-} (e_\delta(x,y)\wedge 1) : x' \leq x\}$
$= 1\text{-}Sup\{e_\delta(x,x')\wedge 1 : x' \leq x\} = 1\text{-}(Sup\{e_\delta(x,x') : x' \leq x\}\wedge 1)$
$= (1\text{-}D(x))\vee 0.$

To prove $a4$), in accordance with (4.4), we have to prove that $Inf\{D(x) : x \in Reg\} = 0$. To this aim, observe that, for any $P \in M$ and $k \in N$, the set $B_k(P) = cl(\{P' \in M\text{: } \delta(P',P) < 1/k\})$ is regular and that $D(B_k(P)) \leq 1/k$. Then $Inf\{D(B_k(P)) : k \in N\} = 0$. The fact that $A4$ is not actually satisfied in an Euclidean space is a consequence of the fact that there is no atom in the class of nonempty regular subsets of such a space. ∎

We call *n-dimensional canonical Euclidean model* the model of graded inclusion space theory defined in the class of nonempty regular bounded subsets of the $n-$dimensional Euclidean space.

5. Witness multi-valued logic and ultrapowers

We call *witness multi-valued logic* any multi-valued logic L such that any interpretation (D, I) of L admits an elementary extension which is ground. We will show that any continuous multi-valued logic is a witness multi-valued logic. To this aim, we must introduce the notion of ultrapower of a fuzzy model. Given a nonempty set S, an *ultrafilter* on S is a class $\mathcal{U}$ of subsets of S such that

i) $X \in \mathcal{U}, Y \in \mathcal{U} \Rightarrow X \cap Y \in \mathcal{U}$
ii) $X \in \mathcal{U}, Y \supseteq X \Rightarrow Y \in \mathcal{U}$
iii) $X \in \mathcal{U} \Leftrightarrow -X \notin \mathcal{U}$.

Obviously, $\emptyset \notin \mathcal{U}$ and $S \in \mathcal{U}$. Moreover any ultrafilter $\mathcal{U}$ satisfies the *finite intersection property*

$$X_1, \ldots, X_n \in \mathcal{U} \Rightarrow X_1 \cap \ldots \cap X_n \in \mathcal{U}$$

and the implication

$$X \cup Y \in \mathcal{U} \Rightarrow \text{either } X \in \mathcal{U} \text{ or } Y \in \mathcal{U}.$$

$\mathcal{U}$ is *principal* if an element $x \in S$ exists such that $\mathcal{U} = \{X : x \in X\}$. If $\mathcal{U}$ is not principal, then there is no finite set in $\mathcal{U}$ and therefore $\mathcal{U}$ contains all the co-finite sets. Given a nonempty set D and an ultrafilter $\mathcal{U}$ on S, we define an equivalence relation $\equiv$ in D^S by setting

$$f \equiv g \Leftrightarrow \{x \in S : f(x) = g(x)\} \in \mathcal{U}.$$

We denote by $[f]_{\mathcal{U}} = \{f' \in D^S : f' \equiv f\}$ the complete class of equivalence defined by an element f in D^S and by $D^S/\mathcal{U} = \{[f]_{\mathcal{U}} : f \in D^S\}$ the quotient of D^S modulo $\equiv$. We write also $[f]$ to denote $[f]_{\mathcal{U}}$. If $d \in D$, then we indicate the class defined by the map constantly equal to d by $[d]$.

Definition 5.1. Let $\mathcal{U}$ be an ultrafilter in S and $f \in [0,1]^S$. We say that l is *a limit of f with respect to* $\mathcal{U}$, and we write $l = lim_{\mathcal{U}} f$ or $l = lim_{\mathcal{U}} f(i)$, provided that $\{i \in S : f(i) \in X\} \in \mathcal{U}$ for every neighborhood X of l.

Such a notion of convergence satisfies the usual properties of the convergence in a topological space. A basic difference is that the limit with respect to an ultrafilter always exists. The following proposition was proved in [1].

Proposition 5.2. *Let* $\mathcal{U}$ *be an ultrafilter in* S *and* $f \in [0,1]^S$*. Then there is exactly one limit of* f *with respect to* $\mathcal{U}$*. If* $\mathcal{U}$ *is a non principal ultrafilter in N, then for any sequence* $(a_n)_{n \in N}$ *of elements in [0,1],*

$$lim_{n \to \infty} a_n = l \Rightarrow lim_{\mathcal{U}} a_n = l. \tag{5.1}$$

Let $g : [0,1]^n \to [0,1]$ *be a continuous function and let* $f_1, ..., f_n$ *be elements in* $[0,1]^S$*, then*

$$g(lim_{\mathcal{U}} f_1(i), ..., lim_{\mathcal{U}} f_n(i)) = lim_{\mathcal{U}} g(f_1(i), ..., f_n(i)). \tag{5.2}$$

Now we are ready to give the notion ultrapower.

Definition 5.3. Let (D, I) be a fuzzy model of a first order language and $\mathcal{U}$ an ultrafilter in a set S. Then *the ultrapower of* (D, I) *modulo* $\mathcal{U}$ is the fuzzy model (D^*, I^*) defined by setting:

$D^* = D^S/\mathcal{U}$

$I^*(g)([f_1], ..., [f_n]) = [(g(f_1(i), ..., f_n(i)))_{i \in S}]$ for any function name g

$I^*(R)([f_1], ..., [f_n]) = lim_{\mathcal{U}} I(R)(f_1(i), ..., f_n(i))$ for any relation name R

$I^*(c) = [I(c)]$ for any constant c.

Notice that for function names and constants this definition coincides with the usual one. This means that, as in the classical logic, for any term t,

$$I^*(t)([f_1], \ldots, [f_n]) = [(I(t)(f_1(i), ..., f_n(i)))_{i \in S}]. \tag{5.3}$$

To prove the main theorem in this section we need the following proposition.

PROPOSITION 5.4. *Let* $(F(i))_{i\in S}$ *be a family of nonempty subsets of* $[0,1]$*, then*

$$Sup\{lim_{\mathcal{U}}s(i) : s \in \Pi_{i\in S}F(i)\} = lim_{\mathcal{U}}(SupF(i)) \tag{5.4}$$

where $\Pi_{i\in S}F(i)$ *is the Cartesian product of* $(F(i))_{i\in S}$.

PROOF. We have to prove that

i) $lim_{\mathcal{U}}(SupF(i))$ is an upper bound of the set $\{lim_{\mathcal{U}}s(i) : s \in \Pi_{i\in S}F(i)\}$;

ii) if l is a *"strict"* upper bound, i.e. $l > lim_{\mathcal{U}}s(i)$ for any $s \in \Pi_{i\in S}F(i)$, then $l \geq lim_{\mathcal{U}}SupF(i))$.

Now, it is evident that for any $s \in \Pi_{i\in S}F(i), lim_{\mathcal{U}}s(i) \leq lim_{\mathcal{U}}(SupF(i))$ and this proves *i*). Let l be such that, $l > lim_{\mathcal{U}}s(i)$, for any $s \in \Pi_{i\in S}F(i)$ and assume, by absurdity, that $l < lim_{\mathcal{U}}(SupF(i))$. Then $X = \{i \in S : l < SupF(i)\} \in \mathcal{U}$. Let s be defined by setting $s(i)$ equal to any element x in $F(i)$ such that $x > l$ in the case $i \in X$ and equal to any element in $F(i)$ otherwise. Then $s \in \Pi_{i\in S}F(i)$ and $\{i \in S : s(i) > l\} \in \mathcal{U}$. This contradicts the fact that the inequality $l > lim_{\mathcal{U}}s(i)$ entails $\{i \in S : s(i) < l\} \in \mathcal{U}$. ■

The following basic theorem was proved in [4].

THEOREM 5.5. *Let* (D, I) *be a fuzzy model of a first order language and* $\mathcal{U}$ *an ultrafilter in a set S. Then, for any formula* α*,*

$$I^*(\alpha, [f_1], \ldots, [f_n]) = lim_{\mathcal{U}}I(\alpha, f_1(i), \ldots, f_n(i)). \tag{5.5}$$

In particular, for any $d_1, ..., d_n$ *in D,*

$$I(\alpha, d_1, \ldots, d_n) = I^*(\alpha, [d_1], \ldots, [d_n]) \tag{5.6}$$

i.e., (D^*, I^*) *is elementary extension of* (D, I).

PROOF. We will prove (5.5) by induction on the complexity of α. In the case α is either the constant **0** or **1**, then (5.5) is evident. Let α be the atomic formula $R(t_1, ..., t_m)$, then

$$\begin{aligned}
&I^*(R(t_1, ..., t_m), [f_1], ..., [f_n]) \\
&= I^*(R)(I^*(t_1)([f_1], ..., [f_n]), ..., I^*(t_m)([f_1], ..., [f_n])) \\
&= I^*(R)([(I(t_1)(f_1(i), ..., f_n(i)))_{i\in S}], ..., [(I(t_m)(f_1(i), ..., f_n(i)))_{i\in S}]) \\
&= lim_{\mathcal{U}}I(R)(I(t_1)(f_1(i), ..., f_n(i), ..., I(t_m)(f_1(i), ..., f_n(i))) \\
&= lim_{\mathcal{U}}I(R(t_1, ..., t_m), f_1(i), \ldots, f_n(i)).
\end{aligned}$$

Assume that (5.5) is satisfied by the formulas $\alpha_1, ..., \alpha_t$ and let h be a logical connective. We clam that (5.5) is satisfied by $h(\alpha_1, ..., \alpha_t)$. Indeed, since the interpretation $\underline{h}$ of h is continuous, by (5.2) we obtain that

$$\begin{aligned}
&I^*(h(\alpha_1,...,\alpha_t),[f_1],...,[f_n])\\
&= \underline{h}(I^*(\alpha_1,[f_1],...,[f_n]),...,I^*(\alpha_t,[f_1],...,[f_n]))\\
&= \underline{h}(lim_{\mathcal{U}} I(\alpha_1, f_1(i),\ldots, f_n(i)),...,lim_{\mathcal{U}} I(\alpha_t, f_1(i),\ldots, f_n(i)))\\
&= lim_{\mathcal{U}}\underline{h}(I(\alpha_1, f_1(i),\ldots, f_n(i)),...,I(\alpha_t, f_1(i),\ldots, f_n(i)))\\
&= lim_{\mathcal{U}} I(h(\alpha_1,...,\alpha_t), f_1(i),\ldots, f_n(i)).
\end{aligned}$$

Assume that (5.5) is satisfied by α, then, by induction hypothesis,

$$\begin{aligned}
&I^*(\exists x_j\alpha,[f_1],...,[f_n])\\
&= Sup\{I^*(\alpha,[f_1],...,[f_{j-1}],[f],[f_{j+1}],...,[f_n]) : f\in D^S\}\\
&= Sup\{lim_{\mathcal{U}} I(\alpha, f_1(i),\ldots, f_{j-1}(i), f(i), f_{j+1}(i),\ldots, f_n(i)) : f\in D^S\}.
\end{aligned}$$

Set, for any $i\in S$,

$$F(i)=\{I(\alpha, f_1(i),\ldots, f_{j-1}(i), f(i), f_{j+1}(i),\ldots, f_n(i)) : f\in D^S\},$$

then it is immediate that

$$F(i)=\{I(\alpha, f_1(i),\ldots, f_{j-1}(i), d, f_{j+1}(i),\ldots, f_n(i)) : d\in D\}$$

and therefore that

$\Pi_{i\in S}F(i) = \{s\in[0,1]^S$: for any $i\in S$ there is $d\in D$ s.t. $s(i) = I(\alpha, f_1(i),\ldots, f_{j-1}(i), d, f_{j+1}(i),\ldots, f_n(i))\}$.

Then, by (5.4),

$$\begin{aligned}
&I^*(\exists x_j\alpha,[f_1],...,[f_n])\\
&= Sup\{lim_{\mathcal{U}} I(\alpha, f_1(i),\ldots, f_{j-1}(i), f(i), f_{j+1}(i),\ldots, f_n(i)) : f\in D^S\}\\
&= Sup\{lim_{\mathcal{U}} s(i) : s\in \Pi_{i\in S}F(i)\}\\
&= lim_{\mathcal{U}}(Sup F(i))\\
&= lim_{\mathcal{U}} Sup\{I(\alpha, f_1(i),\ldots, f_{j-1}(i), d, f_{j+1}(i),\ldots, f_n(i)) : d\in D\}\\
&= lim_{\mathcal{U}} I(\exists x_j\alpha, f_1(i),\ldots, f_n(i)).
\end{aligned}$$

So, (5.5) is satisfied by the formula $\exists x_j\alpha$, too. ■

We conclude this section with the following theorem. Then notion of axiomatizable multi-valued logic is given in [12] (see also [6] and [2]).

Theorem 5.6. *Any continuous logic is a witness multi-valued logic. As a consequence, any axiomatizable logic is a witness logic.*

Proof. Let (D, I) be a fuzzy model of a first order language, $\mathcal{U}$ an ultrafilter in N and (D^*, I^*) the ultrapower of (D, I) modulo $\mathcal{U}$. We claim that (D^*, I^*) is a ground model. Indeed, let $\alpha(x_1,...,x_n)$ be any formula whose only free variable is x_j and assume that $I^*(\exists x_j\alpha) = 1$. Then, by (5.6), we have that $I(\exists x_j\alpha(x_j)) = 1$ and therefore $Sup_{d\in D} I(\alpha, d) = 1$. This entails that a sequence $(d_i)_{i\in N}$ of elements in D exists such that $lim_{i\to\infty}I(\alpha, d_i) = 1$. Then, by (5.5), $I^*(\alpha, [(d_i)_{i\in N}]) = lim_{\mathcal{U}}I(\alpha, d_i) = lim_{n\to\infty}I(\alpha, d_i) = 1$. This proves that (D^*, I^*) is a ground model. Finally, since in [2] was proven that a multi-valued logic is a axiomatizable if and only if it is continuous, any axiomatizable multi-valued logic is a witness logic. ■

6. Whitehead's abstraction processes

The solution proposed in Section 5 is elegant and universal in nature. Unfortunately it is rather unsatisfactory from the point of view we are interested in, i.e. to represent the passage from potential into actual existence. As an example, let *HUGE* be the model of wide set theory defined in Proposition 3.2 and let $HUGE^*$ be its ultrapower modulo an ultra-filter $\mathcal{U}$ in N. Also, consider the sequence

$$I_n = [n(n-1)/2, n(n+1)/2]$$

of pairwise disjoint intervals in N. Then, since $I(INF)(I_n) = n/(n+1)$, such a sequence enables us to prove that $\exists x INF(x)$ is satisfied in *HUGE* in a potential way. Assume that $i = [(I_n)_{n\in N}]$, then we have that

$$I^*(INF)(i) = lim_{\mathcal{U}}I(INF)(I_n) = lim_{n\to\infty}n/(n+1) = 1,$$

i.e. i is an element of $HUGE^*$ satisfying the predicate *INF*. Unfortunately i is a very pathological "infinite set" since there is no (standard) element in it. Indeed, given any $m \in N$, the set $\{n \in N : m \in I_n\}$ contains only an element and therefore $\{n \in N : m \in I_n\} \notin \mathcal{U}$.

A similar argument can be formulated in the theory of graded inclusion spaces. Indeed, let *GI* be the 1-dimensional canonical Euclidean model and define a sequence $(r_n)_{n\in N}$ of real numbers by setting $r_1 = 1$ and $r_{n+1} = r_n + 1/n$. Also, denote by I_n the closed interval $[r_n, r_{n+1}]$. Then since

$$p(I_n) = (1 - D(I_n)) \vee 0 = n/(n+1),$$

such a sequence enables us to claim that $\exists x PNT(x)$ is potentially satisfied in *GI*. Again, consider the ultrapower GI^* of *GI* modulo $\mathcal{U}$ and let i be the complete equivalence class $[(I_n)_{n\in N}]$. Then

$$I^*(PNT)(i) = lim_{n\to\infty}n/(n+1) = 1,$$

and therefore that i satisfies the predicate *PNT* in GI^*. On the other hand, the "point" i is rather unusual since, due to the fact that $\{n \in N\colon\ x \supseteq I_n\}$ is either empty or finite, no (standard and bounded) region x contains i. This is very far from our intuition and from Whitehead's definition of the "abstraction processes". In fact, in Whitehead the process leading from a sequence $(d_n)_{n\in N}$ of "concrete" objects to an "abstract" object d is based on the idea that d_{n+1} and d_n are different approximations of d and that d_{n+1} is "more close" to d than d_n.

Now, we can represent this only by assuming that some kind of order is defined in the domain D. On the other hand, in both the proposed canonical examples there is an order relation playing a basic role. In fact, if we interpret a "big" finite set x as an approximation of an infinite set (an abstract object), then a finite set y containing x is an improvement of such an approximation. Likewise, if we interpret a "small" region x as an approximation of a point (an abstract object), then a region y contained in x is an improvement of such an approximation. In both the case we write $y \preceq x$ to denote such a situation. This suggests the following definition extending the definition given by Whitehead in a geometrical setting.

DEFINITION 6.1. Given an order relation $\preceq$ in a set D we call *abstraction process* any sequence $(x_n)_{n\in N}$ of elements of D such that

$$n \leq m \Rightarrow x_m \preceq x_n.$$

We denote the class of the abstraction processes by $D^{\preceq}$.

Notice that Whitehead requires also that no region x exists such that $x \preceq x_n$ for every $n \in N$. We skip out this condition since in such a way the structures we define are an extension of the ordered set $(D, \preceq)$. In fact, every element $x \in D$ is associated with the abstraction process defined by the sequence constantly equal to x. In order to combine the notion of abstraction process with the one of ultra-product, let (D, I) be a fuzzy model with an order relation $\preceq$ in D. Then we say that an operation g is *compatible* with $\preceq$ provided that it is order-preserving, i.e.

$$d_1 \preceq d'_1, ..., d_n \preceq d'_n \Rightarrow I(g)(d_1, ..., d_n) \preceq I(g)(d'_1, ..., d'_n).$$

A predicate R is *compatible* with $\preceq$, provided that it is order-reversing, i.e.

$$d_1 \preceq d'_1, ..., d_n \preceq d'_n \Rightarrow I(R)(d_1, ..., d_n) \geq I(R)(d'_1, ..., d'_n).$$

Notice that both the graded inclusion in GI and the graded equipotence in $HUGE$ are not compatible. In spite of that, we can reformulate these models by focusing our attention only on the vague predicates PNT and INF we assume as primitive.

If g is a compatible operation and $f_1, ..., f_p$ are abstraction processes, then the sequence $(I(g)(f_1(n), ..., f_p(n)))_{n\in N}$ is an abstraction process, too. This justifies the following definition.

Definition 6.2. Let (D, I) be a fuzzy interpretation of a language $\boldsymbol{L}$ and $\preceq$ an order relation in D. Assume that all the predicates and operations are compatible with respect to $\preceq$. Then, given an ultra-filter $\mathcal{U}$, we denote by $(D^*_{\preceq}, I^*)$ the fuzzy substructure of (D^*, I^*) defined by the subset

$$D^*_{\preceq} = \{[(x_n)_{n\in N}] \in D^* : (x_n)_{n\in N} \text{ is an abstraction process}\}$$

In accordance with (5.6), the map associating every $d \in D$ with $[d]$, is an embedding of (D, I) into $(D^*_{\preceq}, I^*)$. If $\preceq$ is the identity relation, then the abstraction processes coincide with the constant sequences and $(D^*_{\preceq}, I^*)$ coincides with (D, I).

7. Equivalence between abstraction processes

Definition 6.2 is still unsatisfactory. Indeed, the sequences

$$([-1/n, 1/n])_{n\in N} \quad ; \quad ([-1/n^2, 1/n^2])_{n\in N}$$

are not equivalent abstraction processes in the one-dimensional canonical model of *GI*. This means that they define two different abstract objects (i.e. two different *points*) in spite of the fact that our intuition suggests that both represent 0. This *"pathology"* is not surprising, obviously, since it is on the basis of non-standard analysis. As a matter of fact, the ultrapower process produces too many abstract objects and we have to try to introduce a further equivalence relation in $D^{\leq}$ to reduce the number of abstract objects. To do this, at first we generalize a basic definition of Whitehead.

Definition 7.1. We say that an abstraction process $(y_n)_{n\in N}$ *W-dominates* an abstraction process $(x_n)_{n\in N}$, in brief $(y_n)_{n\in N} \geq_W (x_n)_{n\in N}$, if for any y_n there is x_m such that $x_n \preceq y_m$. We say that $(y_n)_{n\in N}$ is *W-equivalent* to $(x_n)_{n\in N}$, in brief $(y_n)_{n\in N} \equiv_W (x_n)_{n\in N}$, provided that $(y_n)_{n\in N} \geq_W (x_n)_{n\in N}$ and $(x_n)_{n\in N} \geq_W (y_n)_{n\in N}$.

The relation $\preceq_W$ is a pre-order in $D^{\preceq}$ and $\equiv_W$ is its associated equivalence relation. Whitehead interprets an element in the corresponding quotient as an abstract geometrical element. We will utilize such an equivalence to modify the notion of ultra-power.

Proposition 7.2. *Let (D, I) be a fuzzy model with a transitive relation $\preceq$ and let $\mathcal{U}$ be a non-principal ultrafilter in N. Then the relation $\equiv_{W\mathcal{U}}$ defined by setting*

$$[(x_n)_{n\in N}] \equiv_{W\mathcal{U}} [(\underline{x}_n)_{n\in N}] \Leftrightarrow (x_n)_{n\in N} \equiv_W (\underline{x}_n)_{n\in N}$$

is a congruence in $D^*_{\preceq}$.

Proof. To prove that $\equiv_{W\mathcal{U}}$ is well defined, we prove that the relation $\equiv_{\mathcal{U}}$ is contained in $\equiv_{W\mathcal{U}}$, i.e. that

$$(x_n)_{n\in N} \equiv_{\mathcal{U}} (y_n)_{n\in N} \Rightarrow (x_n)_{n\in N} \equiv_W (y_n)_{n\in N}.$$

Indeed, given any y_h, since $\{n \in N : x_n = y_n\} \in \mathcal{U}$ and $\{n \in N : y_h \succeq y_n\}$ is cofinite, the set $\{n \in N : y_h \succeq y_n\} \cap \{n \in N : x_n = y_n\}$ belongs to $\mathcal{U}$ and therefore is nonempty. Then $n \in N$ exists such that $y_h \succeq y_n = x_n$. This proves that $(y_n)_{n\in N}$ dominates $(x_n)_{n\in N}$. In the same way we prove that $(x_n)_{n\in N}$ dominates $(y_n)_{n\in N}$ and this shows that $(x_n)_{n\in N} \equiv_W (y_n)_{n\in N}$.

To prove that $\equiv_{W\mathcal{U}}$ is a congruence, let $[f_1], ..., [f_p]$ and $[g_1], ..., [g_p]$ be elements in $D^*_{\preceq}$ such that $[f_1] \equiv_{W\mathcal{U}} [g_1], ..., [f_p] \equiv_{W\mathcal{U}} [g_p]$. Then, given any relation symbol R in $\boldsymbol{L}_{\preceq}$, since $f_1, ..., f_p$ and $g_1, ..., g_p$ are abstraction processes, the sequences $(I(R)(f_1(n), ..., f_p(n)))_{n\in N}$ and $(I(R)(g_1(n), ..., g_p(n)))_{n\in N}$ are order-preserving. This entails the existence of $lim_{n\to\infty}I(R)(f_1(n), ..., f_p(n))$ and $lim_{n\to\infty}I(R)(g_1(n), ..., g_p(n))$. Moreover,

$$\begin{aligned}&I^*(R)([f_1], ..., [f_p])\\ &= lim_{\mathcal{U}}I(R)(f_1(n), ..., f_p(n))\\ &= lim_{n\to\infty}I(R)(f_1(n), ..., f_p(n))\end{aligned}$$

and

$$\begin{aligned}&I^*(R)([g_1], ..., [g_p])\\ &= lim_{\mathcal{U}}I(R)(g_1(n), ..., g_p(n))\\ &= lim_{n\to\infty}I(R)(g_1(n), ..., g_p(n)).\end{aligned}$$

Since each f_i dominates g_i, we have that, for any $n \in N$, there is $m \in N$ such that $f_1(n) \succeq g_1(m), ..., f_p(n) \succeq g_p(m)$ and therefore

$$I(R)(f_1(n), ..., f_p(n)) \leq I(R)(g_1(m), ..., g_p(m)).$$

Consequently,

$$lim_{n\to\infty}I(R)(f_1(n), ..., f_p(n)) \leq lim_{n\to\infty}I(R)(g_1(n), ..., g_p(n)).$$

In a similar way one proves that

$$lim_{n\to\infty}I(R)(g_1(n), ..., g_p(n)) \leq lim_{n\to\infty}I(R)(f_1(n), ..., f_p(n)).$$

Thus,

$$\begin{aligned}&I^*(R)([f_1] ,...,[f_p])\\ &= lim_{n\to\infty}I(R)(f_1(n), ..., f_p(n))\\ &= lim_{n\to\infty}I(R)(g_1(n), ..., g_p(n))\\ &= I^*(R)([g_1], ..., [g_p]).\end{aligned}$$

Let g be an operation and, given $n \in N$, let $m \in N$ such that $f_1(n) \succeq g_1(m), ..., f_p(n) \succeq g_p(m)$. Then

$$I(g)(f_1(n), ..., f_p(n)) \succeq I(g)(g_1(m), ..., g_p(m)).$$

This proves that

$$(I(g)(f_1(n), ..., f_p(n)))_{n\in N} \geq_W (I(g)(g_1(n), ..., g_p(n)))_{n\in N}.$$

In a similar way one proves that

$$(I(g)(g_1(n), ..., g_p(n)))_{n\in N} \geq_W (I(g)(f_1(n), ..., f_p(n)))_{n\in N}.$$

Thus

$$I(g)(f_1(n), ..., f_p(n)))_{n\in N} \equiv_W I(g)(g_1(n), ..., g_p(n)))_{n\in N}. \qquad \blacksquare$$

Definition 7.3. We denote by (D^*_W, I^*) the quotient of $(D^*_{\preceq}, I^*)$ modulo $\equiv_{W\mathcal{U}}$ and we call $\preceq$-*ultrapower* of (D, I) such a model.

It is not clear whether the proposed notion of $\preceq$-ultrapower is satisfactory or not with respect to the question we are interested in, i.e. the potential and actual existence. In any case it seems to give a rather general extension of the techniques used in graded inclusion space theory and in wide set theory.

8. Some questions

How to select the order relation? We have to be careful in such a choice. As an example, consider the sequences $([-1/n, 0])_{n\in N}$ and $([0, 1/n])_{n\in N}$ of *regions* in the canonical model of point-free geometry in the real number set. These processes are not equivalent and therefore they define two different *"points"* which we denote by -0 and +0, respectively. On the other hand $([-1/n, 1/n])_{n\in N}$ is a third abstraction process defining a new point which we denote by 0. It is evident that $0 > -0$ and $0 > +0$ and that -0 and +0 are not comparable. Again, the proposed method creates too many points for people interested in restating the usual mathematica entities. This is the reason why Whitehead in *Process and Reality* refers to the *non-tangential inclusion* $\ll$, topological in nature, whose interpretation in the canonical models is that $x \ll y$ provided that x is contained in the interior of y. This eliminates sequences as $([-1/n, 0])_{n\in N}$ and $([0, 1/n])_{n\in N}$ (note that in our formalism we have to set $\preceq$ equal to the reflexive extension of $\ll$).

Another question is whether the compatibility hypothesis is too restrictive or not. Now, in spite of the appearance, there is a large class of models satisfying such an hypothesis. Indeed we can consider the possibility of managing fuzzy models with incomplete information about the elements of the domain D under consideration. This means that we have to refer to a new domain D' whose elements are pieces of information on the elements in D. Moreover, we have to assume that these pieces of information are ordered with respect to a relation $x \preceq y$ whose meaning is that *"the information x extends the information y"*. More precisely, if we assume that a piece of information on an element of D is a subset of D and if we admit the possibility of a *"fusion"* of different pieces of information, then we have to assume that D' is a closure system. In accordance, given a fuzzy model (D, I), we can consider the fuzzy model (D', I') defined by assuming that:

- D' is a closure system in D,
- $I'(g)(X_1, \ldots, X_n)$ is the element in D' generated by the set $\{I(g)(x_1, \ldots, x_n) \in D : x_1 \in X_1, \ldots, x_n \in X_n\}$,
- $I'(R)(X_1, \ldots, X_n) = Inf\{I'(R)(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}$,

where $X_1, \ldots, X_n$ are in D'. Moreover, we assume that a vague monadic predicate C exists representing the completeness of the information. If we denote by $\preceq$ the inclusion relation, then it is natural to assume that the interpretation of C is order-reversing with respect to $\preceq$ and therefore compatible. Then, in the so obtained fuzzy interpretation all the predicates and operations are compatible. As an example, interval analysis originates in this way from the field of real numbers, by assuming that D' is the class of nonempty closed intervals and by interpreting C through the fuzzy subset s defined by setting $s([a, b]) = 10^{a-b}$.

Finally, a basic question is whether it is correct to use classical mathematics to give a model for the notion of potentiality or not. Indeed, classical mathematics deals with actual objects only and it looks to be contradictory using actual objects to define objects whose existence is claimed to be potential! Nevertheless this is not too surprising if we look at the history of mathematics. As an example, all the models of non Euclidean geometry where defined on the inside of Euclidean geometry. So, if we consider non-Euclidean geometry as a philosophy, it is contradictory to base it on Euclidean geometry, a totally different philosophy. Instead, if non-Euclidean geometry is only a mathematical theory, then the discovery of these models is only a proof of its consistence once we admit the consistence of the Euclidean geometry. The same argumentation holds true for the models of intuitionistic logic which are defined inside classical set theory (as an example by the Kripke-style semantics or by the Heyting algebras). On account of these considerations, the task of defining the notion of potential existence by starting from a mathematics in which only the actual existence is admitted is perhaps acceptable. In fact, our aim is not to define a philosophy which is in an alternative with classical mathematics. Also, perhaps we can accept that the potential existence is related to a level of our mental processes and that it becomes actual existence in a successive level.

In any case these questions are very hard and far from the aim and the ambition of this paper. In fact the paper can be simply taken as a contribution to the theory of witnessed models in fuzzy logic and, in a sense, the concept of "potential existence" can be completely avoided.

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