Preface

Fuzzy logic in narrow sense is a promising new chapter of formal logic whose basic ideas were formulated by Lotfi Zadeh (see Zadeh [1975]a). The aim of this theory is to formalize the *"approximate reasoning"* we use in everyday life, the object of investigation being the human aptitude to manage vague properties (as, for example, *"beautiful"*, *"small"*, *"plausible"*, *"believable"*, etc.) that by their own nature can be satisfied to a degree different from 0 (false) and 1 (true). It is worth noting that the traditional deductive framework in many-valued logic is different from the one adopted in this book for fuzzy logic: in the former logics one always uses a *"crisp"* deduction apparatus, producing crisp sets of formulas, the formulas that are considered logically valid. By contrast, fuzzy logical deductive machinery is devised to produce a fuzzy set of formulas (the theorems) from a fuzzy set of formulas (the hypotheses). Approximate reasoning has generated a very interesting literature in recent years. However, in spite of several basic results, in our opinion, we are still far from a satisfactory setting of this very hard and mysterious subject.

The aim of this book is to furnish some theoretical devices and to sketch a general framework for fuzzy logic. This is also in accordance with the non-Fregean attitude of the book. Indeed, we aim to give some instruments to define rough mathematical models of the wonderful human capacity of reasoning with vague notions and not to propose a unique rigorous formalized (multivalued) logic able to settle such human activity. Consequently, in the book there is no definitive choice of the logical connectives and of their associated interpretations. Also, our constant usage of the set of truth values given by the unit real interval [0,1], is only due to the purpose of simplifying the treatment of the material and not to the conviction that different sets of truth-values are useless. In any case, the book is mainly an exposition of our ideas and research and it does not have any pretension of completeness. In particular, we do not expose the very important fuzzy logic related to Łukasiewicz truth-functional logic. Very good books and papers exist on this argument and we strongly suggest their direct reading (see, for example, Pavelka [1979]c, Cignoli, D'ottaviano and Mundici [2000], Häjek [1998], Novák, Perfilieva and Mockor [1999], Turunen [1999]).

Mainly, three tools are proposed and examined:

- the theory of fuzzy closure operators,
- an extension principle for closure operators,
- the theory of recursively enumerable fuzzy subsets.

Indeed, we embrace Tarski's viewpoint, according to which a monotone logic is a set (of formulas) together with a closure operator (the deduction operator). Consequently, in Chapters 1 and 2 the theory of closure operators in a lattice is exposed and applied to outline an abstract approach to fuzzy logic. In Chapters 3 and 6 an extension principle for classical closure operators is also proposed and largely used. This principle enables us to extend any crisp logic into a fuzzy logic. In Chapter 7 such an approach is generalized by showing that it is possible to associate any chain of crisp logics with a fuzzy logic. In such a way we obtain a very interesting class of fuzzy logics we call *"stratified"*. Necessity logic, graded consequence operator theory and similarity logic all belong to this class. Chapter 4 introduces the notion of fuzzy inferential apparatus in a *"Hilbert style"* by giving a fuzzy set of logical axioms and a suitable set of fuzzy inference rules. In particular, we prove that this approach is equivalent to the theory of continuous fuzzy closure systems (see Theorem 2.6). In Chapter 5 we extend the definition of approximate reasoning usually proposed in the literature by assuming that a deduction apparatus is a tool to calculate constraints on the possible truth values of the formulas. Truth-functional multi-valued logic is examined in Chapter 8 where a strong connection between axiomatizability and continuity of the logical connectives is established (see Theorems 4.5 and Theorem 5.5). In Chapter 9 we propose fuzzy logics probabilistic in nature which are strictly related to super-additive probabilities, upper-lower probabilities, lower envelopes and belief measures. In Chapter 10 we present a tentative approach to fuzzy control by translating any system of IF-THEN fuzzy rules into a system of fuzzy clauses, i.e., a fuzzy program. This enables us to unify the treatment based on the triangular norms and the treatment based on the implications. Finally, in Chapter 11 we extend to fuzzy sets the fundamental notions of decidability and recursive enumerability. In accordance, a definition of an effective Hilbert system is proposed and compared with a definition of enumeration fuzzy operator. This will enable us to get a constructive version of the just quoted Theorem 2.6 (see Theorem 9.2). Also, the notion of recursively enumerable fuzzy subset enables us to obtain several interesting limitative results for fuzzy logic.

Note that three types of fuzzy logics are considered. The first type arises from a *fuzzification of the metalogic*. Necessity logic, similarity logic and graded consequence theory, are typical examples. Indeed, these logics are obtained by admitting that the notions of hypothesis, identity between formulas, logical consequence, can be vague. In this case, the worlds we will describe are *"crisp"*; vagueness arises from the language and the deductive apparatus we use. The second type is related to truth-functional multivalued logic. In such a case *fuzzy worlds* are considered, i.e., worlds whose elements can have graded properties. Finally, we define logics, probabilistic in nature, which are related to *"belief measures"*. Perhaps, we can allocate these logics to the first class. Indeed, in this case vagueness concerns the metalogic notion of *"believable"*.

We expect the book to be read by people interested in artificial intelligence, fuzzy control, formal logic, philosophy. The book is almost completely self-contained but some familiarity with classical logic is required. Moreover, Chapter 11 assumes some acquaintance with the theory of recursive functions.

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