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Probability-Like Functionals and Fuzzy Logic*

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The aim of this paper is to show that fuzzy logic is a suitable tool to manage several types of probability-like functionals. Namely, we show that the super-additive functions, the necessities, the upper and lower probabilities, and the envelopes can be considered theories of suitable fuzzy logics. Some general results about the compactness in fuzzy logic are also obtained.

1. INTRODUCTION

In [9, 10] fuzzy logic is proposed as a tool for probability logic. Indeed, a fuzzy logic is defined whose models are the finitely additive probabilities and whose theories are the lower envelopes. In this paper we extend such results to probability-like functionals. Namely, we examine fuzzy logics whose semantics are:

—the class M_n of the necessities
—the class M_{sa} of the constant sum super-additive measures,
—the class M_{ul} of the upper-lower probabilities
—the class M_p of the finitely additive probabilities.

The related classes of theories coincide with

—the class of the super-additive measures,
—the class of the necessities
—the class of the upper-lower probabilities
—the class of the lower envelopes,

respectively.

Some general result about the compactness in fuzzy logic is also exposed.

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In the sequel we denote by $\mathbb{N}$ and $\mathbb{R}$ the natural number set and the real number set, respectively. Given a set $\mathbb{F}$, a fuzzy subset of $\mathbb{F}$ is any element of the direct power $[0,1]^{\mathbb{F}}$, i.e., any map $s: \mathbb{F} \to [0,1]$. The class of the fuzzy subsets of $\mathbb{F}$ inherits the structure of complete lattice by $[0,1]$ and it is an extension of the lattice of the subsets of $\mathbb{F}$. Indeed if we call *crisp* a subset such that $s(x) \in \{0,1\}$ for every $x \in \mathbb{F}$, then we can identify the subsets of $\mathbb{F}$ with the crisp fuzzy subsets of $\mathbb{F}$ via the characteristic functions. We extend to the lattice $[0,1]^{\mathbb{F}}$ the terminology of set theory; for example, if $s' \leq s$, i.e., $s'(x) \leq s(x)$ for every $x \in F$, then we write $s' \subseteq s$ and we say that s' is *enclosed* in s or that s' is a *part* of s. We call *union* the join operation, and *intersection* the meet operation. The complement $-s$ of a fuzzy subset s is defined by setting $(-s)(x) = 1 - s(x)$ for every $x \in \mathbb{F}$. We say that a fuzzy subset s is *finite* if its support $\mathrm{Supp}(s) = \{x \in \mathbb{F} \mid s(x) = 0\}$ is finite. Also, if $\lambda \in [0,1]$, then we denote by s^{λ} the fuzzy subset constantly equal to λ. So, s^0 denotes the (characteristic function of the) empty set and s^1 the (characteristic function of the) whole set $\mathbb{F}$ of formulas.

Recall that a (classical) *closure operator* in the class $\{0,1\}^{\mathbb{F}} = \boldsymbol{P}(S)$ of subsets of $\mathbb{F}$ is a map J: $\boldsymbol{P}(\mathbb{F}) \to \boldsymbol{P}(\mathbb{F})$ such that, for every X and Y subsets of $\mathbb{F}$.

$$\text{(i) } X \subseteq Y \Rightarrow J(X) \subseteq J(Y); \quad \text{(ii) } X \subseteq J(X); \quad \text{(iii) } J(J(X)) = J(X).$$

A collection $\boldsymbol{C}$ of subsets of $\mathbb{F}$ is *a closure system* if the intersection of any family of element of $\boldsymbol{C}$ is an element of $\boldsymbol{C}$. In particular, since $\mathbb{F}$ is the intersection of the empty family, $\mathbb{F} \in \boldsymbol{C}$. The extension of such concepts to fuzzy set theory is straightforward. We call *fuzzy operator*, in brief *operator*, any map J from $[0,1]^{\mathbb{F}}$ to $[0,1]^{\mathbb{F}}$ and we say that J is a *fuzzy closure operator*, in brief a *closure operator*, provided that

$$\text{(i) } s \subseteq s' \Rightarrow J(s) \subseteq J(s'); \qquad \text{(ii) } s \subseteq J(s); \qquad \text{(iii) } J(J(s)) = J(s).$$

Likewise, a class $\boldsymbol{C}$ of fuzzy subsets of $\mathbb{F}$ is called a *fuzzy closure system*, in brief a *closure system*, if the intersection of any family of elements of $\boldsymbol{C}$ is an element of $\boldsymbol{C}$. Now, it is well known that if J is a closure operator, then the set $\boldsymbol{C}_J = \{X \mid J(X) = X\}$ of fixed points of J is a closure system. Also, if $\boldsymbol{C}$ is any class of subsets, then by setting $J_{\boldsymbol{C}}(X) = \bigcap\{Y \in \boldsymbol{C} \mid Y \supseteq X\}$ we obtain a closure operator $J_{\boldsymbol{C}}$ we call *the closure operator associated with* $\boldsymbol{C}$. It is immediate that such a connection holds for the fuzzy closure operators and the fuzzy closure systems, too. So, if J is a fuzzy operator satisfying (i) and (ii) and

$$\boldsymbol{C}_J = \left\{f \in [0,1]^{\mathbb{F}} \mid J(f) = f\right\},$$

then C_J is a closure system. Also, if C is a class of fuzzy subsets, then the operator J_C defined by

$$J_C(s) = \bigcap \{ s' \in C \mid s' \supseteq s \}$$

is a fuzzy closure operator.

2. FUZZY LOGIC

A fuzzy logic is defined by a fuzzy semantics and a fuzzy syntax as follows. Let $\mathbb{F}$ be a set whose elements are called *formulas*. Then a fuzzy semantics is any class M of fuzzy subsets of $\mathbb{F}$ such that $s^1 \notin M$. A *fuzzy system of axioms* or *initial valuation* is any fuzzy subset of formulas. We say that an element m of M is a *model of a fuzzy system of axioms* v and we write $m \vDash v$ provided that $v \subseteq m$. We call *satisfiable* any fuzzy system of axioms admitting a model in M. As an example, in multivalued sentential calculus M is the class of the valuations of the formulas; in classical first order logic, we can set M equal to (the characteristic functions of) the complete theories. In accordance with the observations at the end of Section 1, a fuzzy semantics M induces a fuzzy closure operator, we call a *consequence operator* and we denote by $C_M : [0,1]^{\mathbb{F}} \to [0,1]^{\mathbb{F}}$, defined by setting, for every fuzzy system of axioms v,

$$C_M(v) = \bigcap \{ m \in M \mid m \vDash v \}.$$

In particular, if v is not satisfiable, then $C_M(v)$ collapses in s^1. We define a *theory* of M as a fixed point of C_M. In particular, s^1 is called *the inconsistent theory* of M. Sometimes we write C instead of C_M.

Note. We can extend the above definitions as follows. An *interval constraint* is a pair (v_1, v_2) of fuzzy subsets such that $v_1 \subseteq v_2$. A *model* of (v_1, v_2) is an element $m \in M$ such that $v_1(\alpha) \leq m(\alpha) \leq v_2(\alpha)$ for every formula α. Instead of the logical consequence operator we consider two operators J and $\hat{J}$ defined as

$$J(v_1, v_2) = \bigcap \{ m \in M \mid v_1 \subseteq m \subseteq v_2 \}$$

and

$$\hat{J}(v_1, v_2) = \bigcup \{ m \in M \mid v_1 \subseteq m \subseteq v_2 \}.$$

In a sense, $J(v_1, v_2)(\alpha)$ expresses a necessity and $\hat{J}(v_1, v_2)(\alpha)$ a possibility. The initial valuations coincide with the interval constraints (v_1, v_2) for which v_2 is constantly equal to 1. In other words, they are lower constraints.

The syntactical apparatus is defined as follows: an n-ary *fuzzy rule of inference* is a pair $r = (r', r'')$ where r' is an n-ary operation defined in a subset $\mathrm{Dom}(r)$ of $\mathbb{F}^n$ and r'' is an n-ary operation on $[0,1]$ preserving joins in each variable. So, an inference rule r consists of a *syntactical component* r' that operates on formulas (in fact, it is a rule of inference in the usual sense) and a *valuation component* r'' that operates on truth values to calculate how the truth value of the conclusion depends on the truth values of the premises. We indicate an application of an inference rule r by

$$\frac{\alpha_1, \dots, \alpha_n}{r'(\alpha_1, \dots, \alpha_n)}; \qquad \frac{\lambda_1, \dots, \lambda_n}{r''(\lambda_1, \dots, \lambda_n)}$$

whose meaning is that *if you know that the formulas* $\alpha_1, \dots, \alpha_n$ *are true at least to the degree* $\lambda_1, \dots, \lambda_n$, *then you can conclude that the formula* $r'(\alpha_1, \dots, \alpha_n)$ *is true at least to the degree* $r''(\lambda_1, \dots, \lambda_n)$.

A *fuzzy syntax* on $\mathbb{F}$ is a pair $\mathcal{S} = (a, R)$ where a is a fuzzy subset of $\mathbb{F}$, the fuzzy subset of *logical axioms*, and R is a set of fuzzy rules of inference. A fuzzy subset s of formulas is *closed with respect to the rule* r if, for every $(\alpha_1, \dots, \alpha_n) \in D(r)$

$$s(r'(\alpha_1, \dots, \alpha_n)) \geq r''(s(\alpha_1), \dots, s(\alpha_n)).$$

A *theory* on the fuzzy syntax $\mathcal{S}$ is a fuzzy subset of formulas containing the fuzzy subset of logical axioms and closed with respect to every rule in R. Obviously, the (characteristic function of the) whole set of formulas $\mathbb{F}$ is a theory that we call the *inconsistent theory*. We say that a theory is *maximal* if it is a maximal element in the class of the consistent theories. A *proof* of a formula α is a sequence $\pi = \alpha_1, \dots, \alpha_m$ of formulas where $\alpha_m = \alpha$, equipped with related "justifications." This means that, for every $i = 1, \dots, m$, we have to specify whether

(i) α_i is assumed as a logical axiom; or

(ii) α_i is assumed as a proper axiom; or

(iii) α_i is obtained by an inference rule.

In the last case we have to indicate also the rule and the formulas in $\alpha_1, \dots, \alpha_{i-1}$ used to obtain α_i. The justifications are necessary in order to evaluate the proofs when a fuzzy subset of axioms is given. Indeed, let $v\colon \mathbb{F} \to [0,1]$ be any initial valuation. Then *the valuation* $\mathrm{Val}(\pi, v)$ of a proof π with respect to v is defined by induction on the length m of π by setting

$$\mathrm{Val}(\pi, v) = \begin{cases} a(\alpha_m) & \text{if } \alpha_m \text{ is assumed as a logical axoim} \\ v(\alpha_m) & \text{if } \alpha_m \text{ is assumed as a proper axiom} \\ r''(\mathrm{Val}(\pi(i_1), v), \ldots, \mathrm{Val}(\pi(i_n), v)) & \text{if } a_m = r'(\alpha_{i_1}, \ldots, \alpha i_n), \end{cases}$$

where, for every $i \leq m$, $\pi(i)$ denotes the proof $\alpha_1, \ldots, \alpha_i$. If α is the formula proven by π, the meaning we assign to $\mathrm{Val}(\pi, v)$ is that *given the information* v, *the proof* π *assures that* α *holds at least to degree* $\mathrm{Val}(\pi, v)$. Now, unlike the usual Hilbert inferential systems, in a fuzzy syntax different proofs of a same formula α can give different contributions to the degree of validity of α. So, in order to evaluate α, we have to refer to the whole set of proofs of α.

DEFINITION 2.1. We define the *deduction operator* as the fuzzy operator $D_S: [0,1]^{\mathbb{F}} \to [0,1]^{\mathbb{F}}$ such that, for every initial valuation v and every formula α,

$$D_S(v)(\alpha) = \mathrm{Sup}\{\mathrm{Val}(\pi, v) \mid \pi \text{ is a proof of } \alpha\}. \tag{2.1}$$

The meaning of $D_S(v)(\alpha)$ is still

given the information v, *we may prove that*

α holds at least at degree $D_S(v)(\alpha)$,

but we have also that

$D_S(v)(\alpha)$ *is the best possible valuation*

we can draw from the information v.

Generally, we write D instead of D_S. We say that $D(v)$ is the *fuzzy subset of consequences* of v. We say also that v is *inconsistent* if $D(v)$ is the inconsistent theory. In [12] J. Pavelka proves the following facts.

PROPOSITION 2.2. *Let* D *be the deduction operator of a fuzzy syntax* S. *Then*

—*the theories of* S *are the fixed points of* D

—*the intersection of a family of theories is a theory*

—*for every system of axioms* v

$$D(v) = \bigcap\{\tau/\tau \text{ is a theory and } \tau \supseteq v\}$$

—D *is a fuzzy closure operator.*

DEFINITION 2.3. A *fuzzy logic* is a pair (M, S) where M is a fuzzy semantics and S a fuzzy syntax such that the logical consequence operator C_M coincides with the deduction operator D_S.

Every fuzzy syntax S determines a fuzzy logic with respect to a suitable fuzzy semantics. Indeed, by Proposition 2.2, it is sufficient to set M equal to the set of consistent theories of S. The proof of the following proposition is obvious.

PROPOSITION 2.4. *(M, S) is a fuzzy logic if and only if every element of M is a theory and every theory of S is a theory of M.*

3. COMPACTNESS

Denote by $\mathrm{Sat}(M)$ the class of satisfiable initial valuations. Then we give two notions of compactness for a fuzzy semantics.

DEFINITION 3.1. A fuzzy semantics M is *compact* if for every initial valuation v,

$$v \in \mathrm{Sat}(M) \Leftrightarrow v_f \in \mathrm{Sat}(M) \qquad \text{for every finite fuzzy subset } v_f. \tag{3.1}$$

M is called *logically compact* if $\mathrm{Sat}(M)$ is inductive, i.e., the union of a directed class of satisfiable initial valuations is a satisfiable initial valuation.

Given two fuzzy subsets s_1 and s_2, we set $s_1 \ll s_2$ provided that $s_1(x) < s_2(x)$ for every $x \in \mathrm{Supp}(s_1)$. The relation $\ll$ enables us to characterize the logical compactness.

PROPOSITION 3.2. *A fuzzy semantics M is logically compact iff*

$$v \in \mathrm{Sat}(M) \Leftrightarrow v_f \in \mathrm{Sat}(M) \qquad \text{for every finite } v_f \ll v. \tag{3.2}$$

Proof. See Murali [11]. ∎

This proposition shows that every logically compact semantics is compact. The converse implication is not true, in general. For example, consider the fuzzy semantics $M = \{s \in \mathfrak{F}(\mathbb{F}) \mid s(x) \neq 1 \text{ for every } x \in \mathbb{F}\}$. Then it is immediate that M is compact. Since s^1 is the limit of the class of satisfiable fuzzy subsets s^λ, $\lambda \neq 1$, M is not logically compact.

PROPOSITION 3.3. *If M is logically compact, then every satisfiable initial valuation admits a maximal model. Equivalently, every element of M is contained in a maximal element in M.*

Proof. Assume that v is satisfiable. Then, since $\mathrm{Sat}(\mathcal{M})$ is inductive, the class $\mathcal{C} = \{s \in \mathrm{Sat}(\mathcal{M}) \mid s \supseteq v\}$ is inductive. By Zorn's Lemma, a maximal element s of $\mathcal{C}$ exists. Since s is satisfiable, $m \in \mathcal{M}$ exists such that $m \supseteq s$. Since $m \in \mathcal{C}$ by the maximality of s we can conclude that $s = m$. This proves both that s belongs to $\mathcal{M}$ and that s is a maximal element in $\mathcal{M}$. ∎

In order to prove a suitable compactness criterion, we recall some elementary concepts of ultraproduct theory. Let $(\lambda_i)_{i \in I}$ be a family of elements of $[0, 1]$ and $\mathcal{U}$ a filter on I. Then we write $\lim_{\mathcal{U}} \lambda_i = \lambda$ provided that

$$\forall \epsilon > 0\ \exists X \in \mathcal{U}\ \forall i \in X \qquad |\lambda - \lambda_i| \leq \epsilon.$$

Equivalently, we can write

$$\text{for every interval } (a, b) \text{ containing } \lambda,\ \{i \in I \mid \lambda_i \in (a, b)\} \in \mathcal{U}.$$

Such a notion of convergence satisfies the same properties of the classical one but in addition, if $\mathcal{U}$ is prime, for any family $(\lambda_i)_{i \in I}$, $\lim_{\mathcal{U}} \lambda_i$ always exists. Also, assume that I is the set $\mathbb{N}$ of natural numbers and that $\mathcal{U}$ is not principal. Then,

$$\lim_{n \to \infty} \lambda_n = \lambda \Rightarrow \lim_{\mathcal{U}} \lambda_n = \lambda$$

(see, e.g., [5, Theorem 1.5.1]). Such notions enable us to define the notion of ultraproduct of a family of fuzzy subsets.

DEFINITION 3.4. Let $(s_i)_{i \in I}$ be a family of fuzzy subsets of $\mathbb{F}$ and $\mathcal{U}$ an ultrafilter on I. Then the *ultraproduct of* $(s_i)_{i \in I}$ *modulo* $\mathcal{U}$ is the fuzzy subset $s\colon \mathbb{F} \to [0, 1]$ defined by setting, for every $x \in \mathbb{F}$,

$$s(x) = \lim_{\mathcal{U}} s_i(x).$$

THEOREM 3.5. *Let $\mathcal{M}$ be closed with respect to the ultraproducts and let v be a fuzzy system of axioms. Then,*

(i) *the ultraproduct of a family of models of v is a model of v*

(ii) *for every formula α a model m of v exists such that*

$$C(v)(\alpha) = m(\alpha)$$

(iii) *$\mathcal{M}$ is logically compact.*

Proof.

(i) This is an immediate consequence of the definition of limit with respect to a filter.

(ii) Let v be an initial valuation and α a formula. Then, since $C(v)(\alpha) = \operatorname{Inf}\{m(\alpha) \mid m \in \mathcal{M}, m \supseteq v\}$, a sequence $(m_n)_{n \in \mathbb{N}}$ of models of v exists such that $m_n(\alpha)$ is a decreasing sequence of numbers such that $C(v)(\alpha) = \lim_{n \to \infty} m_n(\alpha)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and m the ultraproduct of $(m_n)_{n \in \mathbb{N}}$ modulo $\mathcal{U}$. Then, m is a model of v such that

$$m(\alpha) = \lim_{\mathcal{U}} m_n(\alpha) = \lim_{n \to \infty} m_n(\alpha) = C(v)(\alpha).$$

(iii) We apply Proposition 3.2. Assume that v_f is satisfiable for every v_f finite such that $v_f \ll v$. At first we prove that every finite fuzzy subset s of v is satisfiable. Indeed, it is easy to find an increasing sequence v_n of finite fuzzy subsets such that $s(x) = \lim_{n \to \infty} v_n(x)$ for every x and $v_n \ll s$. Since we have also that $v_n \ll v$, by hypothesis a sequence of models m_n exists such that $m_n \supseteq v_n$. Let $\mathcal{U}$ be a non-principal prime filter and let m be the ultraproduct of the sequence $(m_n)_{n \in \mathbb{N}}$ modulo $\mathcal{U}$. Then, since

$$m(\alpha) = \lim_{\mathcal{U}} m_n(\alpha) \geq \lim_{\mathcal{U}} v_n(\alpha) = \lim_{n \to \infty} v_n(x) = s(\alpha)$$

we have that m is a model of v.

Denote by I the class of finite subsets of $\mathbb{F}$ and let $i \in I$. Then, since the restriction of v to i is satisfiable, an element m_i of $\mathcal{M}$ exists such that $m_i(x) \geq v(x)$ for every $x \in i$. We find a model m of v as a suitable ultraproduct of the so obtained family $(m_i)_{i \in I}$. To this purpose, we have to find an ultrafilter $\mathcal{U}$ such that for every $x \in \mathbb{F}$ the set $B(x) = \{i \in I \mid m_i(x) \geq v(x)\} \in \mathcal{U}$. In turn, this is equivalent to saying that the class $\{B(x) \mid x \in \mathbb{F}\}$ of subsets of I satisfies the finite intersection property. Now, let $x_1, \dots, x_n$ be formulas and $i = \{x_1, \dots, x_n\}$. Then $m_i(x_j) \geq v(x_j)$ for $j = 1, \dots, n$ and therefore i belongs to $B(x_1) \cap \cdots \cap B(x_n)$. This concludes the proof. ∎

Theorem 3.5 suggests the following general method to obtain logically compact fuzzy semantics (in the next sections the interest of such a method will be apparent). We define a *closed k-ary relation* as a closed subset $\mathcal{R}$ of $\mathbb{R}^k$. The identity and the order relation are examples of closed binary relations. As usual, if $x_1, \dots, x_k$ are real numbers, we write $\mathcal{R}(x_1, \dots, x_k)$ to denote that $(x_1, \dots, x_k) \in \mathcal{R}$.

PROPOSITION 3.6. *Denote by* $\mathcal{M}$ *the class of fuzzy subsets* m *of* $\mathbb{F}$ *satisfying a set of conditions like*

$$\Re\big(m(p_0(x_1, \dots, x_h)), \dots, m(p_k(x_1, \dots, x_h))\big), \tag{3.3}$$

where

—$p_0, \ldots, p_k$ *are partial operations on* $\mathbb{F}$ *defined in a domain* $D \subseteq \mathbb{F}^h$;
—$R \subseteq \mathbb{R}^{k+1}$ *is a closed relation.*

Then M *is closed with respect to the ultraproducts. Consequently, if* $s^1 \notin M$, *then* M *is a logically compact semantics.*

Proof. At first observe that, since R is closed, if $(\lambda_i^0)_{i \in I}, \ldots, (\lambda_i^k)_{i \in I}$ are sequences of real numbers such that $R(\lambda_i^0, \ldots, \lambda_i^k)$ for every $i \in I$, then $R(\lim_U \lambda_i^0, \ldots, \lim_U \lambda_i^k)$. Indeed, set $\lambda^0 = \lim_U \lambda_i^0, \ldots, \lambda^k = \lim_U \lambda_i^k$ and assume that $(\lambda^0, \ldots, \lambda^k)$ is not in R. Then, since the complement of R is open, $I_0, \ldots, I_k$ exist such that $I_0, \ldots, I_k$ are intervals such that $I_0 \times \cdots \times I_k$ is disjoint from $\Re$ and $\lambda^0 \in I_0, \ldots, \lambda^k \in I_k$. As a consequence, the sets

$$X_0 = \{i \in I \mid \lambda_i^0 \in I_0\}, \ldots, X_k = \{i \in I \mid \lambda_i^k \in I_k\}$$

belong to U. Since U is a filter $X_0 \cap \cdots \cap X_k$ is nonempty, so, if j is any element of this intersection, then $(\lambda_j^0, \ldots, \lambda_j^k) \in I_0 \times \cdots \times I_k$. Thus, we have that $(\lambda_j^0, \ldots, \lambda_j^k) \notin R$ and this contradicts the hypothesis.

Now, let $(m_i)_{i \in I}$ be a family of elements of M, U an ultrafilter on I, and m the ultraproduct of $(m_i)_{i \in I}$ by U. Then, since for every $i \in I$,

$$R(m_i(p_0(x_1, \ldots, x_h)), \ldots, m_i(p_k(x_1, \ldots, x_h))),$$

in view of the property we have just proved

$$R\Big(\lim_U (m_i(p_0(x_1, \ldots, x_h))), \ldots, \lim_U (m_i(p_k(x_1, \ldots, x_h)))\Big)$$

and therefore, $m \in M$. ∎

4. FUZZY LOGIC IN A BOOLEAN ALGEBRA

The logics we consider in this paper are strictly related with the classical logic. So, we assume that the set of formulas is a Boolean algebra **B** whose minimum and maximum we denote by **0** and **1**, respectively. As an example, **B** could be the Lindenbaum algebra of a logic, an algebra of events, and so on. Also, we assume that the fuzzy set of logical axioms coincides with the tautologies of classical logic, i.e., we set $a(\mathbf{1}) = 1$ and $a(x) = 0$ for every $x \neq \mathbf{1}$. Obviously, a fuzzy subset s of formulas contains a if and only if $s(\mathbf{1}) = 1$. The presence of a *negation* among the connectives enables us to give some further interesting definitions.

DEFINITION 4.1. If v is a fuzzy subset of formulas, then we denote by $v^{\perp}$ the fuzzy subset defined by setting $v^{\perp}(\alpha) = 1 - v(-\alpha)$ for every $\alpha \in \mathbf{B}$. Also, we define the fuzzy operator $C^{\perp}$ by setting $C^{\perp}(v) = (C(v))^{\perp}$.

Observe that while $C(v)(\alpha)$ is the truth degree of the claim "α is a consequence of v," the number $C^{\perp}(v)(\alpha)$ is the truth degree of "$-\alpha$ is not a consequence of v," i.e., the degree of consistence of α with v. In other words, given a fuzzy information v, $C(v)$ is the fuzzy set of formulas that are necessary and $C^{\perp}(v)$ the fuzzy set of formulas that are possible. This enables us to obtain, for every formula α, an interval approximation $[C(v)(\alpha), C^{\perp}(v)(\alpha)]$ of the actual truth degree of α. If $v' \supseteq v$ (the information increases), then for every formula α the interval $[C(v')(\alpha), C^{\perp}(v')(\alpha)]$ is contained in $[C(v)(\alpha), C^{\perp}(v)(\alpha)]$ and we have more precise information.

DEFINITION 4.2. Let v be an initial valuation and α a formula. Then we say that α is *decidable* in v if $C(v)(\alpha) = C^{\perp}(v)(\alpha)$, i.e., $C(v)(\alpha) + C(v)(-\alpha) = 1$. If every formula is decidable in v, i.e., $C(v) = C^{\perp}(v)$, then v is called *complete*.

Obviously, every complete theory is a maximal theory.

DEFINITION 4.3. A fuzzy semantics is *balanced* provided that all the models are complete. A fuzzy logic is *balanced* if its semantics is balanced.

The following proposition shows that for balanced semantics the expressive power of the interval constraints is the same as the one of the initial valuations (i.e., of the lower constraints).

PROPOSITION 4.4. *Assume that* M *is balanced. Then, given an initial valuation* v

$$m \text{ is a model of } v \Leftrightarrow m \text{ satisfies } (v, v^{\perp}).$$

Given an interval constraint (l, u),

$$m \text{ satisfies } (l, u) \Rightarrow m \text{ is a model of } l \cup u^{\perp}.$$

Proof. This is obvious. ∎

PROPOSITION 4.5. *Assume that* M *is balanced and closed with respect to the untraproducts and let* v *be a satisfiable initial valuation. Then, for every formula* α, *two models* $m, m' \in \mathsf{M}$ *of* v *exist such that*

$$C(v)(\alpha) = m(\alpha) \quad \text{and} \quad C^{\perp}(v)(\alpha) = m'(\alpha).$$

Proof. Equality $C(v)(\alpha) = m(\alpha)$ follows from Theorem 3.5. Let m' be a model of v such that $m'(-\alpha) = C(v)(-\alpha)$. Then

$$m'(\alpha) = 1 - m'(-\alpha) = 1 - C(v)(-\alpha) = C^{\perp}(v)(\alpha). \quad ∎$$

In order to complete a consistent theory, in classical logic one defines the extension T_α of a set T of formulas via a formula α by setting $T_\alpha = T$ if α is inconsistent with T and $T_\alpha = Y \cup \{\alpha\}$ otherwise. Obviously, if T is consistent then T_α is consistent and α is decidible in T_α. In order to extend this notion to fuzzy logics, given a fuzzy set v of formulas we call an *extension of v by α* the fuzzy set v_α defined by setting $v_\alpha = v$ if $1 - v(-\alpha) < v(\alpha)$ and

$$v_\alpha(x) = \begin{cases} v(x) & \text{if } x \neq \alpha \\ v^\perp(\alpha) & \text{if } x = \alpha \end{cases}$$

otherwise.

PROPOSITION 4.6. *Assume that M is a balanced fuzzy semantics closed with respect to the ultraproducts, let τ be a consistent theory, and α a formula. Then τ_α is a consistent theory and α is decidable in τ_α.*

Proof. If τ is satisfiable, then a model $m \in M$ exists such that

$$1 - \tau(-\alpha) \geq 1 - m(-\alpha) \geq m(\alpha) \geq \tau(\alpha).$$

Then $\tau_\alpha(\alpha) = \tau^\perp(\alpha)$ and by Proposition 4.5 a model m' of τ exists such that $m'(\alpha) = \tau^\perp(\alpha) = \tau_\alpha(\alpha)$. Then, m' is a model of τ_α, too. It is immediate that α is decidable in τ_α. ∎

THEOREM 4.7. *Assume that M is balanced and closed with respect to the ultraproducts, let v be a satisfiable valuation, and let $(\alpha_n)_{n \in \mathbb{N}}$ be an enumeration of all the formulas in $\mathbb{F}$. Then by setting*

$$\tau_0 = v; \qquad \tau_{n+1} = C\big((\tau_n)_{\alpha_{n+1}}\big); \qquad m = \bigcup_{n \in \mathbb{N}} \tau_n$$

we obtain a complete theory extending τ (i.e., a model of τ).

Proof. By Proposition 4.6 every τ_n is satisfiable, so, since M is logically compact, m is satisfiable. It is immediate that every formula is decidable in m. ∎

5. NECESSITY LOGIC

At first we consider a non-balanced fuzzy logic that is related with the necessity measures. Since this logic is well known in the literature we confine ourselves only to sketch some results and definitions (see [3, 4, 8]). Recall that a fuzzy subset $n: \mathbf{B} \to [0,1]$ of $\mathbf{B}$ is a *necessity* if $n(\mathbf{1}) = 1$, $n(\mathbf{0}) = 0$, and

$$n(\alpha_1 \wedge \alpha_2) = n(\alpha_1) \wedge n(\alpha_2) \tag{5.1}$$

for every $\alpha_1, \alpha_2 \in \mathbf{B}$. The necessities are a basic tool in fuzzy set theory (see, for example, [7]). We denote by M_n the class of necessities and we consider M_n as a fuzzy semantics.

PROPOSITION 5.1. *The class* M_n *of the necessities is a fuzzy semantics closed with respect to the ultraproducts. Consequently,* M *is logically compact and every necessity is contained in a maximal necessity.*

Proof. We apply Proposition 3.6. Namely, condition $n(\mathbf{1}) = 1$ can be obtained by assuming that $k = 0$, p_0 is the map constantly equal to $\mathbf{1}$, and R the relation $\lambda = 0$. In a similar way we obtain condition $n(\mathbf{0}) = 0$. Equality (5.1) can be obtained by assuming that $p_0(\alpha_1, \alpha_2) = \alpha_1 \wedge \alpha_2$, $p_1(\alpha_1, \alpha_2) = \alpha_1$, $p_2(\alpha_1, \alpha_2) = \alpha_2$, and that R is the relation $\lambda_0 = \lambda_1 \wedge \lambda_2$. ∎

Also we define a fuzzy syntax S_n, we call *n-syntax*, by two rules. The first one is the *collapsing rule* asserting that if we have proven two disjoint formulas α_1 and α_2 at degrees λ_1 and λ_2 and $\lambda_1 + \lambda_2 > 1$, then we may prove the contradiction (at degree 1) too. Thus the whole theory collapses (this is a "control" rule rather than an inference rule). More formally, the collapsing rule is the pair $c = (c', c'')$ where c' is defined in $\{(\alpha_1, \alpha_2) \mid \alpha_1 \wedge \alpha_2 = \mathbf{0}\}$ by setting $c'(\alpha_1, \alpha_2) = \mathbf{0}$ and c'' is the map defined by setting $c''(\lambda_1, \lambda_2) = 1$ if $\lambda_1 + \lambda_2 > 1$ and $c''(\lambda_1, \lambda_2) = 0$ otherwise. Obviously, a fuzzy set of formulas τ is closed with respect to c if and only if the existence of a pair of disjoint formulas α_1 and α_2 such that $\tau(\alpha_1) + \tau(\alpha_2) > 1$ entails that $\tau(\mathbf{0}) = 1$ and therefore (if τ is increasing) that τ is the inconsistent theory. The second fuzzy rule is the following generalization $s = (s', s'')$ of the Modus Ponens,

$$s'(\alpha_1 \to \alpha_2, \alpha_1) = \alpha_2, \qquad s''(\lambda_1, \lambda_2) = \lambda_1 \wedge \lambda_2. \tag{5.2}$$

Then, by assuming that the set of logical axioms is $\{\mathbf{1}\}$, a fuzzy theory of such a syntax is a fuzzy subset τ of formulas such that

(i) $\tau(\mathbf{1}) = 1$

(ii) $\tau(\alpha_2) \geq \tau(\alpha_1 \to \alpha_2) \wedge \tau(\alpha_1)$

(iii) $\tau(\mathbf{0}) = 1$ if α_1 and α_2 exist such that $\tau(\alpha_1) + \tau(\alpha_2) > 1$ and $\alpha_1 \wedge \alpha_2 = \mathbf{0}$.

PROPOSITION 5.2. *A map n is a necessity if and only if it is a consistent theory of the n-syntax. Consequently,* (M_n, S_n) *is a fuzzy logic we call n-logic* or *logic of the necessities*.

Obviously, given an initial valuation v, the necessity $C(v)$ generated by v can be obtained by (2.1). The following proposition, given in [3], indicates a more direct way to obtain $C(v)$. We write $\alpha \to \beta$ to denote that $\alpha \leq \beta$.

PROPOSITION 5.3. *A fuzzy subset of formulas v is consistent if and only if its support* Supp(v) *satisfies the finite intersection property, i.e.*,

$$\alpha_1, \ldots, \alpha_n \in \operatorname{Supp}(v) \Rightarrow \alpha_1 \wedge \cdots \wedge \alpha_n \neq \mathbf{0}.$$

Also, if v is consistent, then

$$D(v)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \mathbf{1} \\ \operatorname{Sup}\{v(\alpha_1) \wedge \cdots \wedge v(\alpha_n) \mid \alpha_1 \wedge \cdots \wedge \alpha_n \to \alpha\} & \text{if } \alpha \neq \mathbf{1}. \end{cases}$$

We conclude this section by noticing that the complete theories of n-logic coincide with the complete "classical" theories, that is, with the ultrafilters of $\mathbf{B}$.

PROPOSITION 5.4. *The following are equivalent*

(i) τ *is maximal*;

(ii) τ *is the characteristic function of an ultrafilter of* $\mathbf{B}$;

(iii) τ *is complete*.

Consequently, every consistent theory is contained in a complete theory.

Proof. (i) ⇒ (ii). Let τ be maximal. Then, since Supp(τ) satisfies the finite intersection property, an ultrafilter U exists containing Supp(τ). Let n be the characteristic function of U. Then n is a necessity extending τ. Since τ is maximal, $\tau = n$ and this proves (ii). (ii) ⇒ (iii) and (iii) ⇒ (i) are obvious. ∎

COROLLARY 5.5. *The class of complete theories is not a semantics for the n-logic. In other words a theory cannot be expressed as an intersection of a family of complete theories, in general.*

6. LOGIC OF THE SUPER-ADDITIVE MEASURES: THE SEMANTICS

The second logic we consider, we call *sa-logic*, admits as a balanced semantics the class M_{sa} of the constant sum super-additive measures. Recall that a *super-additive measure* is a map $p: \mathbf{B} \to [0, 1]$ such that $p(\mathbf{1}) = 1$ and, for every $\alpha, \beta \in \mathbf{B}$ such that $\alpha \wedge \beta = \mathbf{0}$,

$$p(\alpha \vee \beta) \geq p(\alpha) + p(\beta). \tag{6.1}$$

A super-additive measure p is said to be a *constant sum* super-additive measure if, for every $\alpha \in \mathbf{B}$,

$$p(\alpha) + p(-\alpha) = 1. \tag{6.2}$$

Every necessity is a super-additive measure. Indeed, if n is a necessity and $\alpha \wedge \beta = \mathbf{0}$, then either $n(\alpha) = 0$ or $n(\beta) = 0$ and since n is order-preserving, (6.1) is satisfied.

Remark. A super-additive measure is also called a *characteristic function* of an n-persons game. Indeed, consider a game with a set P of players and let $\mathbf{B} = \mathcal{P}(P)$. Every set $\alpha \in \mathbf{B}$ of players is interpreted as a *coalition* in the game and the number $p(\alpha)$ represents the (sure) gain of the coalition provided we set equal to 1 the gain of the whole set of players. The meaning of (6.1) is that the gain of $\alpha \vee \beta$ is greater than or equal to the sum of the gain of α and the gain of β. Indeed, among the possible strategies of the coalition $\alpha \vee \beta$ there are the strategies in which α and β play separately. In general we have that $p(\alpha \vee \beta) > p(\alpha) + p(\beta)$ and this makes the coalition $\alpha \vee \beta$ convenient, while, if p is additive, the coalitions are useless. Also, the constant sum condition means that the game is strictly competitive. An example of constant sum super-additive measure is furnished by a game in which the majority always wins. More specifically, if P is a finite set (the set of players) whose number of elements is odd, $\mathbf{B} = \mathcal{P}(P)$ and p is defined by setting, for every subset α of P

$$p(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ has more elements than } -\alpha \\ 0 & \text{otherwise,} \end{cases}$$

then p is a constant sum super-additive measure while, since $p(\{x\}) = 0$ for every x, p is not a probability.

In the following proposition we give some properties of $\mathcal{M}_{sa}$.

PROPOSITION 6.1. *The class $\mathcal{M}_{sa}$ of the constant sum super-additive functions is a balanced fuzzy semantics closed with respect to the ultraproducts.*

Proof. $\mathcal{M}_{sa}$ is balanced by definition. To prove that $\mathcal{M}_{sa}$ is closed with respect to the ultraproducts, we can apply Proposition 3.6. In fact, we can express the super additivity by setting $\mathcal{R} = \{(\lambda_0, \lambda_1, \lambda_2) \mid \lambda_0 \geq \lambda_1 + \lambda_2\}$, $p_0(x, y) = x \vee y$ with $D = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid x \wedge y = \mathbf{0}\}$, $p_1(x, y) = x$, and $p_2(x, y) = y$. We can express the constant sum condition by setting $\mathcal{R} = \{(\lambda_0, \lambda_1) \mid \lambda_0 + \lambda_1 = 1\}$, $p_0(x) = x$, $p_1(x) = -x$. ∎

7. LOGIC OF THE SUPER ADDITIVE MEASURES: THE FUZZY SYNTAX

The fuzzy syntax S_{sa}, we call the *sa-syntax,* is defined by adding to the collapsing rule the *disjunction rule* given by

$$\frac{\alpha_1, \alpha_2}{\alpha_1 \vee \alpha_2}, \qquad \frac{\lambda_1, \lambda_2}{\lambda_1 \oplus \lambda_2} \; (\alpha_1 \wedge \alpha_2 = \mathbf{0}),$$

where $\oplus$ is the Lukasievicz sum defined by setting $x \oplus y = 1 \wedge (x + y)$ for every x and y in $[0, 1]$. More formally, the disjunction rule is the fuzzy rule $d = (d', d'')$ such that

$$\mathrm{Dom}(d) = \{(\alpha_1, \alpha_2) \in \mathbf{B} \times \mathbf{B} \mid \alpha_1 \wedge \alpha_2 = \mathbf{0}\};$$
$$d'(\alpha_1, \alpha_2) = \alpha_1 \vee \alpha_2;$$
$$d''(\lambda_1, \lambda_2) = \lambda_1 \oplus \lambda_2.$$

The proofs in this syntax are very simple. As an example, if α is the join of the disjoint formulas $\alpha_1, \alpha_2, \alpha_3$, then the following picture represents a proof π of α such that $\mathrm{Val}(\pi, v)$ is the number $v(\alpha_1) \oplus v(\alpha_2) \oplus v(\alpha_3)$

$$\frac{\dfrac{\alpha_1, \alpha_2}{\alpha_1 \vee \alpha_2}, \alpha_3}{\alpha}; \qquad \frac{\dfrac{v(\alpha_1), v(\alpha_2)}{v(\alpha_1) \oplus v(\alpha_2)}, v(\alpha_3)}{v(\alpha_1) \oplus v(\alpha_2) \oplus v(\alpha_3)}.$$

THEOREM 7.1. *The consistent theories in S_{sa} coincide with the super-additive measures. As a consequence, M_{sa} is the class of the consistent complete theories of S_{sa}.*

Proof. Let τ be a consistent theory of the sa-syntax. Then $\tau(\mathbf{1}) = 1$. To prove that τ satisfies (6.1), let α and β be two incompatible formulas. Then by the disjunction rule $\tau(\alpha \vee \beta) \geq \tau(\alpha) \oplus \tau(\beta)$. In the case $\tau(\alpha \vee \beta) < 1$, since $\tau(\alpha) \oplus \tau(\beta) = \tau(\alpha) + \tau(\beta)$, this inequality coincides with (6.1). If $\tau(\alpha \vee \beta) = 1$ and, by absurdity $\tau(\alpha) + \tau(\beta) > 1$, then by the collapsing rule

$$\tau(\mathbf{0}) = \tau(c'(\alpha, \beta)) \geq c''(\tau(\alpha), \tau(\beta)) = 1.$$

Since this contradicts the consistence of τ, we may conclude that $\tau(\alpha) + \tau(\beta) \leq 1 = \tau(\alpha \vee \beta)$. Conversely, it is immediate that every super-additive measure is a consistent theory of S_{sa}. ∎

Note that, since every necessity is a super additive measure, in a sense, the sa-logic is an extension of the logic of the necessities. The next proposition gives a simple way to obtain the theory $D(v)$ generated by an initial valuation v.

PROPOSITION 7.2. *A fuzzy set of formulas v is consistent iff,*

$$v(\alpha_1) + \cdots + v(\alpha_n) \leq 1 \tag{7.1}$$

for every sequence $\alpha_1, \ldots, \alpha_n$ of pairwise disjoint formulas. If this is the case we have that $D(v)(\mathbf{1}) = 1$ and, if $\alpha \neq \mathbf{1}$,

$$D(v)(\alpha) = \operatorname{Sup}\{v(\alpha_1) + \cdots + v(\alpha_n) \mid \alpha_1 \vee \cdots \vee \alpha_n \to \alpha,\ \alpha_i \wedge \alpha_j = \mathbf{0} \textit{ for } i \neq j\}. \tag{7.2}$$

Proof. Assume (7.1) and define $\bar{v}$ by setting $\bar{v}(\mathbf{1}) = 1$ and

$$\bar{v}(\alpha) = \operatorname{Sup}\{v(\alpha_1) + \cdots + v(n) \mid \alpha_1 \vee \cdots \vee \alpha_n \to \alpha,\ \alpha_i \wedge \alpha_j = \mathbf{0}\ \forall i \neq j\} \quad \text{if } \alpha \neq \mathbf{1}.$$

Then (7.1) entails that the values of $\bar{v}$ are in $[0, 1]$. Also, for every pair α and β of disjoint formulas

$$\begin{aligned}
\bar{v}(\alpha \vee \beta) &= \operatorname{Sup}\{v(\alpha_1) + \cdots + v(\alpha_n) \mid \alpha_1 \vee \cdots \vee \alpha_n \to \alpha \vee \beta,\\
&\qquad \alpha_i \wedge \alpha_j = \mathbf{0}\ \forall i \neq j\}\\
&\geq \operatorname{Sup}\{v(\alpha_1) + \cdots + v(\alpha_n) \mid \text{either } \alpha_1 \vee \cdots \vee \alpha_n \to \alpha\\
&\qquad \text{or } \alpha_1 \vee \cdots \vee \alpha_n \to \beta,\ \alpha_i \wedge \alpha_j = \mathbf{0}\ \forall i \neq j\}\\
&\geq \operatorname{Sup}\{v(\alpha_1) + \cdots + v(\alpha_n) \mid \alpha_1 \vee \cdots \vee \alpha_n \to \alpha,\\
&\qquad \alpha_i \wedge \alpha_j = \mathbf{0}\ \forall i \neq j\}\\
&\quad + \operatorname{Sup}\{v(\alpha_1) + \cdots + v(\alpha_n) \mid \alpha_1 \vee \cdots \vee \alpha_n \to \beta,\\
&\qquad \alpha_i \wedge \alpha_j = \mathbf{0}\ \forall i \neq j\}\\
&= \bar{v}(\alpha) + \bar{v}(\beta).
\end{aligned}$$

Then $\bar{v}$ is a super-additive function. It is immediate that $\bar{v} \supseteq v$. Let p be a super-additive measure such that $p \supseteq v$. Then, for every α and $\alpha_1, \ldots, \alpha_n$ pairwise disjoint formulas such that $\alpha_1 \vee \cdots \vee \alpha_n \to \alpha$, we have that $v(\alpha_1) + \cdots + v(\alpha_n) \leq p(\alpha_1) + \cdots + p(\alpha_n)$ and hence that $\bar{v}(\alpha) \leq p(\alpha)$.

This proves that $\bar{v}$ is the intersection of the super-additive function containing v and, hence, that $\bar{v} = D(v)$. Obviously, this entails also the consistence of v. Conversely, if v is consistent, then the theory $D(v)$ is a super-additive measure containing v and therefore

$$v(\alpha_1) + \cdots + v(\alpha_n) \leq D(v)(\alpha_1) + \cdots + D(v)(\alpha_n)$$
$$\leq D(v)(\alpha_1 \vee \cdots \vee \alpha_n) \leq 1. \quad \blacksquare$$

8. THE COMPLETENESS THEOREM

In accordance with Theorem 7.1, the class of super-additive measures is a fuzzy semantics for the syntax S_{sa}. Since this class is closed with respect to the ultraproducts, we have the following proposition.

PROPOSITION 8.1. *The class of consistent theories of S_{sa} is logically compact.*

Now, we prove a theorem analogous to Theorem 4.6.

PROPOSITION 8.2. *Let τ be a consistent theory of S_{sa}. Then, given any formula α,*

—*the extension τ_α of τ by α is consistent*

—$D(\tau_\alpha)(-\alpha) = \tau(-\alpha)$ *and* $D(\tau_\alpha)(\alpha) = 1 - \tau(-\alpha)$

—*α is decidable in τ_α.*

Proof. Since by hypothesis τ satisfies (7.1), we have that $\tau(\alpha) + \tau(-\alpha) \leq 1$ and therefore $\tau_\alpha(\alpha) = 1 - \tau(-\alpha)$. To prove that τ_α satisfies (7.1) we may confine ourselves to the case in which $\alpha_1, \dots, \alpha_n$ are pairwise disjoint formulas such that $\alpha_1 = \alpha$. Now, since $-\alpha \geq \alpha_2 \vee \cdots \vee \alpha_n$, we have that $\tau(-\alpha) \geq \tau(\alpha_2 \vee \cdots \vee \alpha_n)$ and hence that

$$\tau_\alpha(\alpha) + \tau_\alpha(\alpha_2) + \cdots + \tau_\alpha(\alpha_n) = 1 - \tau(-\alpha) + \tau(\alpha_2) + \cdots + \tau(\alpha_n)$$
$$\leq 1 - \tau(-\alpha) + \tau(\alpha_2 \vee \cdots \vee \alpha_n) \leq 1.$$

This proves the consistence of τ_α. Also, since $D(\tau_\alpha) \supseteq \tau_\alpha \supseteq \tau$,

$$1 \geq D(\tau_\alpha)(-\alpha) + D(\tau_\alpha)(\alpha) \geq \tau_\alpha(-\alpha) + (\tau_\alpha)(\alpha)$$
$$= \tau(-\alpha) + 1 - \tau(-\alpha) = 1$$

and this proves the remaining part of the proposition. $\blacksquare$

The following proposition is analogous to Theorem 4.7.

PROPOSITION 8.3. *Let τ be a consistent theory, α a formula, and $(\alpha_n)_{n\in\mathbb{N}}$ an enumeration of the formulas in* **B** *such that $\alpha_1 = -\alpha$. Then by setting*

$$\tau_0 = \tau; \qquad \tau_{n+1} = D((\tau_n)\alpha_{n+1}); \qquad m = \bigcup_{n\in\mathbb{N}} \tau_n$$

we obtain a satisfiable complete theory extending τ such that

$$m(\alpha) = \tau(\alpha) \qquad \text{and} \qquad m(-\alpha) = 1 - \tau(\alpha).$$

Proof. By Proposition 8.2 every τ_n is consistent. By Proposition 8.1, m is consistent. The remaining part of the proposition is immediate. ∎

Note that there is no difficulty to extend the above theorems to the case in which **B** is not enumerable. Indeed, the proof works well also by assuming that **B** is well ordered.

THEOREM 8.4. *Every consistent theory is an intersection of a family of complete theories.*

Proof. Let τ be a consistent theory and α a formula. Then to prove the proposition it is sufficient to observe that Proposition 8.3 entails that for every formula α a complete theory m exists such that $m \supseteq \tau$ and $m(\alpha) = \tau(\alpha)$. ∎

As an immediate consequence, the following completeness theorem holds.

THEOREM 8.5. *The sa-syntax S_{sa} together with the semantics M_{sa} of the constant sum super-additive measures defines a fuzzy logic we call the* logic of super-additive measures (*in brief* sa-logic).

Proof. By Theorem 7.1 every element of M_{sa} is a consistent theory of S_{sa}. By Theorem 8.4 every theory τ of S_{sa} is an intersection of elements of M_{sa}. ∎

9. LOGIC OF THE UPPER-LOWER PROBABILITIES

Now we consider an extension of the logic of the super-additive measures whose fuzzy semantics is the class M_{ul} of the upper-lower probabilities. Recall that an *upper-lower probability* is a super-additive function p such that, for every $\alpha, \beta \in$ **B**

$$p^{\perp}(\alpha \vee \beta) \leq p^{\perp}(\alpha) + p^{\perp}(\beta) \tag{9.1}$$

or, equivalently

$$p(\alpha \wedge \beta) \geq p(\alpha) + p(\beta) - 1. \tag{9.2}$$

(As a matter of fact, in the literature the name upper-lower probability is attributed to the pair $(p, p^{\perp})$, see, e.g., [6].) In the remark in Section 2 an example of n-person game was exposed whose characteristic function is a super additive measure that is not an upper-lower probability.

EXAMPLES. Let S be a finite set, $p\colon \boldsymbol{F}(S) \to [0,1]$ a frequency measure, and $f\colon S \to \mathbb{R}$ a map (a random variable). As an example, S is a set of possible experiments and $f(x)$ is a physical quantity measured in x. Then we can define a map μ by setting, for every measurable subset X of $\mathbb{R}$,

$$\mu(X) = p(\{x \in S \mid f(x) \in X\}). \tag{9.3}$$

In other words $\mu(X)$ is the frequency (equivalently, the probability) of the experiments in which the physical quantity $f(x)$ satisfies the condition X. It is well known that the so defined function is a probability. Now, assume that we are not able to indicate the precise value of f. Then we may substitute f with a multivalued function $\tilde{f}$ in such a way that, for every $x \in S$, $\tilde{f}(x)$ is an interval-constraint on the actual value $f(x)$. In this case, it is very natural to set

$$\mu(X) = p\left(\left\{x \in S \mid \tilde{f}(x) \subseteq X\right\}\right)$$

and the number $\mu(X)$ is the frequency of the experiments x in which we are sure that $f(x)$ satisfies X. We have that *the function μ above defined is an upper and lower probability. Such a function is not a probability, in general.*

Indeed, if X and Y are two disjoint subsets of $\mathbb{R}$, then

$$\begin{aligned}
\mu(X \cup Y) &= p\left(\left\{x \in S \mid \tilde{f}(x) \subseteq X \cup Y\right\}\right) \\
&\geq p\left(\left\{x \in S \mid \tilde{f}(x) \subseteq X\right\} \cup \left\{x \in S \mid \tilde{f}(x) \subseteq Y\right\}\right) \\
&= p\left(\left\{x \in S \mid \tilde{f}(x) \subseteq X\right\}\right) + p\left(\left\{x \in S \mid \tilde{f}(x) \subseteq Y\right\}\right) \\
&= \mu(X) + \mu(Y).
\end{aligned}$$

One similarly proves that $\mu(X \cap Y) \geq \mu(X) + \mu(Y) - 1$. To prove that μ is not a probability, observe that it may be that $\{x \in S \mid \tilde{f}(x) \subseteq X\} \cup \{x \in S \mid \tilde{f}(x) \subseteq -X\} \neq S$ and therefore that $\mu(X) + \mu(-X) \neq 1$.

PROPOSITION 9.1. *The class M_{ul} of the upper-lower probabilities is a fuzzy semantics closed with respect to the ultraproducts.*

Proof. To prove that M_{ul} is closed with respect to the ultraproducts we can apply Proposition 3.6 by setting $R_3 = \{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1 \geq \lambda_2 + \lambda_3 - 1\}$, $p_0(x, y) = x \wedge y$, $p_1(x, y) = x$, $p_2(x, y) = y$. ∎

To axiomatize M_{ul}, we define the *upper-lower syntax*, in brief *ul-syntax*, as the extension S_{ul} of S_{sa} obtained by adding the *conjunction rule* defined by

$$\frac{\alpha_1, \alpha_2}{\alpha_1 \wedge \alpha_2}, \qquad \frac{\lambda_1, \lambda_2}{\lambda_1 \odot \lambda_2},$$

where $\odot$ is the Lukasiewicz conjunction, i.e., $x \odot y = 0 \vee (x + y - 1)$ for every x and y in $[0, 1]$. More precisely, the conjunction rule $c = (c', c'')$ is defined by setting

$$c'(\alpha, \beta) = \alpha \wedge \beta; \qquad c''(x, y) = x \odot y.$$

To give an example of a proof in S_{ul}, let $\alpha = (\alpha_1 \vee \alpha_2) \wedge \alpha_3$ where $\alpha_1 \wedge \alpha_2 = \mathbf{0}$. Then the following picture

$$\frac{\dfrac{\alpha_1 \alpha_2}{\alpha_1 \vee \alpha_2} \quad \alpha_3}{\alpha = (\alpha_1 \vee \alpha_2) \wedge \alpha_3}; \qquad \frac{\dfrac{v(\alpha_1) v(\alpha_2)}{v(\alpha_1) \oplus v(\alpha_2)} \quad v(\alpha_3)}{(v(\alpha_1) \oplus v(\alpha_2)) \odot v(\alpha_3)}$$

represents a proof of α whose valuation is the number $(v(\alpha_1) \oplus v(\alpha_2)) \odot v(\alpha_3)$.

PROPOSITION 9.2. *For every fuzzy set of formulas p,*

$$p \text{ is a consistent theory of } S_{ul} \Leftrightarrow p \text{ is an ul-probability}$$

$$p \text{ is a consistent complete theory of } S_{ul} \Leftrightarrow p \text{ is a probability.}$$

Proof. It is sufficient to note that the closure with respect to the conjunction rule is equivalent to (9.2). ∎

PROPOSITION 9.3. *The ul-syntax S_{ul}, together with the fuzzy semantics M_{ul} defines a logically compact fuzzy logic extending the sa-logic.*

Proof. This is immediate. ∎

PROPOSITION 9.4. *The balanced elements of M_{ul} coincide with the probabilities. Consequently, no balanced semantics equivalent to M_{ul} exists.*

Proof. Let τ be a balanced element of M_{ul}. Then since $\tau = \tau^{\perp}$, from (6.1) and (9.1) we have that τ is a probability. It is immediate that every probability is a balanced element of M_{ul}. Assume that M is a balanced semantics (i.e., a class of probabilities) equivalent to M_{ul}. Then every element of M_{ul} is the intersection of probabilities and therefore a lower envelope (see Section 10). This is an absurdity since, as it is well known, there are upper and lower probabilities that are not envelopes. ∎

Note that, while it is immediate that every probability is a maximal theory, I do not know if the converse is true. Moreover, given a fuzzy set v of formulas, by Proposition 9.3 the *ul*-probability generated by v coincides with the theory $D(v)$ in the syntax S_{ul} and it may be obtained as usual by the formula $D(v)(\alpha) = \mathrm{Sup}\{\mathrm{Val}(\pi, v) \mid \pi$ is a proof of $\alpha\}$. Unlike the logic of the necessities and the logic of the super-additive measures, I don't know a more simpler formula to obtain $D(v)$.

10. LOGIC OF THE ENVELOPES: THE SEMANTICS

The last (and more important) fuzzy logic we consider is an extension of the logic of the upper-lower probabilities we call *logic of the envelopes*. I considered this logic in [9] extensively, so in this paper I confine myself only to expose some definitions and results. The semantics under consideration is the class M_p of the finitely additive probabilities in **B**. This choice is rather natural since a probability represents a complete information about a random phenomena and therefore a complete theory. In this case the information furnished by an initial valuation v is that, for every formula α, "the probability of α is at least $v(\alpha)$." In accordance, the models of v are the probabilities greater than or equal to v. Moreover, the probabilistic theory $C(v)$ generated by v is the least upper bound of the probabilities greater than or equal to v. Since, in literature, any map that may be obtained as the least upper bound (i.e., the intersection) of a family of probabilities is named a *lower envelope*, in brief *envelope*, we can say also that $C(v)$ is the envelope "generated" by v. The semantics M_p is balanced, so we have that

—*to give a fuzzy set of axioms v is equivalent to giving an interval constraint for the unknown probability distribution*,

—*the operator C enables us to improve the initial interval constraints, more specifically, to give the best interval constraints given v.*

This means that the search for a fuzzy syntax fitting well this semantics is not only a theoretical task but, as a matter of fact, it is a search for algorithms able to improve initial probabilistic valuations. Such a problem is strictly related with the construction of diagnostic systems and, for example, was examined by Weichselberger and Pöhlmann in [14] under the assumption that **B** is finite and that only the atoms of **B** are valued.

To give a suitable condition of consistence we have to introduce a new family of connectives in **B**. Set $\mathbb{N}_0 = \mathbb{N} - \{0\}$, let $h \in \mathbb{N}$, and $k \in \mathbb{N}_0$. Then we have that the *h-k-connective* is the h-ary operation C^k on **B** defined by setting $C^0(\alpha_1, \dots, \alpha_h) = 1$ and, in the case $k \neq 0$,

$$C^k(\alpha_1, \dots, \alpha_h) = \bigvee \left\{ \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \mid i_1, \dots, i_k \text{ are distinct} \right\}.$$

It is immediate that

$$C^0(\alpha_1, \dots, \alpha_h) \geq C^1(\alpha_1, \dots, \alpha_h) \geq \cdots \geq C^h(\alpha_1, \dots, \alpha_h),$$

$$C^1(\alpha_1, \dots, \alpha_h) = \alpha_1 \vee \cdots \vee \alpha_h, C^h(\alpha_1, \dots, \alpha_h) = \alpha_1 \wedge \cdots \wedge \alpha_h,$$

$$C^{h+1}(\alpha_1, \dots, \alpha_h) = C^{h+2}(\alpha_1, \dots, \alpha_h) = \cdots = \mathbf{0},$$

$$C^k(\alpha_1, \dots, \alpha_h) \geq C^k(\alpha_1, \dots, \alpha_{h-1}),$$

$$C^k(\alpha_1, \dots, \alpha_h) = C^k(\alpha_1, \dots, \alpha_h, \mathbf{0}) = C^{k+1}(\alpha_1, \dots, \alpha_h, \mathbf{1}).$$

The connection between the h-k-connectives and the probabilities is expressed by the following proposition.

PROPOSITION 10.1. *A map p such that $p(\mathbf{1}) = 1$ and $p(\mathbf{0}) = 0$ is a probability iff, for every $\alpha_1, \dots, \alpha_h$ in* **B**,

$$p(\alpha_1) + \cdots + p(\alpha_h) = p\left(C^1(\alpha_1, \dots, \alpha_h)\right) + \cdots + p\left(C^h(\alpha_1, \dots, \alpha_h)\right). \tag{10.1}$$

Given the formulas $\alpha_1, \dots, \alpha_h$, we set

$$M(\alpha_1, \dots, \alpha_h) = \operatorname{Max}\left\{k \in \mathbb{N} \mid C^k(\alpha_1, \dots, \alpha_h) \neq \mathbf{0}\right\}, \tag{10.2}$$

or, equivalently,

$$M(\alpha_1, \dots, \alpha_h) = \operatorname{Min}\left\{k \in \mathbb{N} \mid C^k(\alpha_1, \dots, \alpha_h) = \mathbf{0}\right\} - 1. \tag{10.3}$$

As it is proven in the following proposition, the function M enables us to characterize the satisfiable fuzzy sets of formulas.

PROPOSITION 10.2. *The semantics M_p of the probabilities is closed with respect to the ultraproducts and therefore logically compact. Moreover, an initial valuation v is satisfiable iff, for $\alpha_1, \dots, \alpha_h$ in* **B**

$$v(\alpha_1) + \cdots + (\alpha_h) \leq M(\alpha_1, \dots, \alpha_h). \tag{10.4}$$

11. LOGIC OF THE ENVELOPES: THE FUZZY SYNTAX

To find suitable inference rules for the logic of the envelopes, we recall at first some characterizations of the envelopes (see [1, 9]).

PROPOSITION 11.1. *Let p be a fuzzy set of formulas such that* $p(\mathbf{1}) = 1$. *Then the following are equivalent*

(a) *p is an envelope*;

(b) $p(\alpha_1) + \cdots + p(\alpha_h) \leq (M(\alpha_1, \ldots, \alpha_h) - k + 1) \cdot p(C^k(\alpha_1, \ldots, \alpha_h)) + k - 1$ *for every* $h \in \mathbb{N}$, $k \in \mathbb{N}_0$, *and* $\alpha_1, \ldots, \alpha_h$ *sequence of formulas*.

Such a proposition suggests the following inference rules. Denote by $[\![\ ]\!]: \mathbb{R} \to [0,1]$ the function defined by setting $[\![n]\!] = 0$ if $x \leq 0$, $[\![x]\!] = 1$ if $x \geq 1$, and $[\![x]\!] = x$ in the interval $[0,1]$. Then, given h, m, k integer numbers such that $h \geq m \geq k$, we define the *h-m-k-rule* of inference as the rule $r = (r', r'')$ defined in

$$D(h,m) = \{(\alpha_1, \ldots, \alpha_h) \mid M(\alpha_1, \ldots, \alpha_h) = m\}$$

by

$$r'(\alpha_1, \ldots, \alpha_h) = C^k(\alpha_1, \ldots, \alpha_h),$$

$$r''(x_1, \ldots, x_h) = [\![\frac{x_1 + \cdots + x_h - k + 1}{m - k + 1}]\!].$$

For example, the h-1-1-rule is defined by

$$\frac{\alpha_1, \ldots, \alpha_h}{\alpha_1 \vee \cdots \vee \alpha_h}; \qquad \frac{\lambda_1, \ldots, \lambda_h}{\lambda_1 \oplus \cdots \oplus \lambda_h},$$

where it is assumed that $\alpha_1, \ldots, \alpha_h$ are pairwise disjoint. By setting $h = 2$ we obtain the disjunction rule in accordance with the fact that the logic of the envelopes is an extension of the logic of the super-additive measures. By setting $h = m = k$ we obtain

$$\frac{\alpha_1, \ldots, \alpha_h}{\alpha_1 \wedge \cdots \wedge \alpha_h}; \qquad \frac{\lambda_1, \ldots, \lambda_h}{\lambda_1 \odot \cdots \odot \lambda_h},$$

where $\alpha_1 \wedge \cdots \wedge \alpha_h \neq \mathbf{0}$. In particular, by setting $h = 2$, we obtain the conjunction rule and this shows that the logic of the envelopes is also an extension of upper-lower probability logic. Note that the class of *h-m-k*-rules is not independent. For instance, the closure with respect to the 2-1-1-rule (i.e., the disjunction rule) entails the closure with respect to any of the above h-1-1-rules. Moreover, the meaning of a h-m-k-rule is not too clear, in general. It is an open question to find a smaller and more intuitive system of rules.

Proposition 10.2 suggests considering, for every h and m, $h \geq m$, the *h-m-collapsing rule* $c = (c', c'')$ defined in $D(h, m)$ by setting $c'(\alpha_1, \dots, \alpha_h) = \mathbf{0}$ for every $\alpha_1, \dots, \alpha_h$ and

$$c''(\lambda_1, \dots, \lambda_h) = \begin{cases} 1 & \text{if } \lambda_1 + \cdots + \lambda_h > m \\ 0 & \text{otherwise}. \end{cases}$$

The h-m-collapsing rule says that if we have proven the formulas $\alpha_1, \dots, \alpha_h$ with degrees $\lambda_1, \dots, \lambda_h$ that violate the consistence condition (10.4), then we can prove the contradiction $\mathbf{0}$ (and therefore any formula). As an example, the 2-1-collapsing rule entails that if α_1 and α_2 are disjoint formulas proven with degree $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{2}{3}$, then we can prove that $\mathbf{0}$ is true.

DEFINITION 11.2. The *probabilistic deduction system* is the fuzzy syntax S_p whose inference rules are the h-m-k-rules and the h-m-collapsing rules. We call any theory of this syntax a *probabilistic theory*.

In the probabilistic deduction system a proof proceeds step by step as follows:

—assume that we have early proven $\alpha_1, \dots, \alpha_h$ with degree at least $\lambda_1, \dots, \lambda_h$

—if $\lambda_1 + \cdots + \lambda_h > M(\alpha_1, \dots, \alpha_h)$, then the available information is inconsistent,

—otherwise, conclude that $C^k(\alpha_1, \dots, \alpha_h)$ holds at least at degree

$$\frac{\lambda_1 + \cdots + \lambda_h - k + 1}{M(\alpha_1, \dots, \alpha_h) - k + 1}.$$

THEOREM 11.3. *The semantics M_p of the finitely additive probabilities and the system S_p define a fuzzy logic we call* the logic of the lower envelopes.

12. SUBSTITUTING THE BOOLEAN ALGEBRAS WITH THE ZERO-ORDER LANGUAGES

We could start from the set $\mathbb{F}$ of formulas of the propositional calculus instead of from a Boolean algebra. If we want to do so, we have to consider only *transparent* models, i.e., valuations $m\colon \mathbb{F} \to U$ of the formulas such that

$$\alpha \equiv \beta \Rightarrow m(\alpha) = m(\beta).$$

As an example, consider the case of the *ul*-logic. Then we define the semantics as the class M of the fuzzy subsets $m: \mathbb{F} \to U$ of $\mathbb{F}$ such that

(i) $m(\alpha) = 1$ for every tautology α

(ii) $\alpha \to \beta$ tautology $\Rightarrow m(\alpha) \le m(\beta)$

(iii) α inconsistent with $\beta \Rightarrow m(\alpha \vee \beta) \ge m(\alpha) + m(\beta)$

(iv) $m(\alpha \wedge \beta) \ge m(\alpha) + m(\beta) - 1.$

The following proposition shows the connection between such a semantics and the upper-lower probabilities.

PROPOSITION 12.1. *The fuzzy subset of formulas $m: \mathbb{F} \to U$ satisfies* (i)–(iv) *iff there exist a Boolean algebra* $\mathbf{B}$, *a Boolean valuation* $Bv: \mathbb{F} \to \mathbf{B}$, *and an upper-lower probability* $p: \mathbf{B} \to U$ *such that* $m = p \circ Bv$.

$$\begin{tikzcd} \mathbb{F} \arrow[r, "Bv"] \arrow[dr, "m"'] & \mathbf{B} \arrow[d, "p"] \\ & U \end{tikzcd}$$

Proof. Assume that $m: \mathbb{F} \to U$ satisfies (i)–(iv) and let $\mathbf{B}$ be the Lindembaum algebra associated with $\mathbb{F}$. Then, the function $Bv: \mathbb{F} \to \mathbf{B}$ defined by setting $Bv(\alpha) = [\alpha]$ is a Boolean valuation. Define $p: \mathbf{B} \to U$ by setting, for every $[\alpha] \in \mathbf{B}$, $p([\alpha]) = m(\alpha)$. By (ii) such a definition is correct. It is immediate that p is an *ul*-probability such that $m(\alpha) = p(Bv(\alpha))$. The converse part is obvious. ∎

A fuzzy deduction apparatus S for M is obtained by assuming the set of tautologies as logical axioms and by considering a disjunction rule

$$\frac{\alpha, \beta}{\alpha \vee \beta}; \qquad \frac{\lambda, \mu}{\lambda \oplus \mu} \ (\alpha \text{ inconsistent with } \beta),$$

a Modus Ponens rule

$$\frac{\alpha, \alpha \to \beta}{\beta}; \qquad \frac{\lambda, \mu}{\lambda \odot \mu},$$

and a collapsing rule

$$\frac{\alpha, \beta}{\alpha \wedge \beta}, \qquad \frac{\lambda, \mu}{c(\lambda, \mu)} \ (\alpha \text{ inconsistent with } \beta),$$

where $c(\lambda, \mu) = 1$ if $\lambda + \mu > 1$ and $c(\lambda, \mu) = 0$ otherwise.

PROPOSITION 12.2. *The class of consistent theories of* S *coincides with the semantics* M. *Consequently*, (M, S) *is a fuzzy logic* (*in Hilbert style*).

Proof. If τ is a consistent theory of S, then (i) and (iv) are immediate. To prove (ii), assume that $\alpha \to \beta$ is a tautology. Then,

$$\tau(\beta) \geq \tau(\alpha) \odot \tau(\alpha \to \beta) = \tau(\alpha) \odot 1 = \tau(\alpha).$$

To prove (iii) it is sufficient to observe that if α is inconsistent with β, then $\tau(\alpha) \oplus \tau(\beta) = \tau(\alpha) + \tau(\beta)$. Otherwise by the collapsing rule $\tau(\alpha \wedge \beta) \geq c(\tau(\alpha), \tau(\beta)) = 1$ and, since $\alpha \wedge \beta \to \gamma$ is a tautology, for every formula γ, $\tau(\gamma) = 1$.

Conversely, assume that (i)–(iv) are satisfies by τ. Then it is immediate that τ contains the set of logical axioms and that τ is closed with respect to the disjunction rule and the collapsing rule. Since $\alpha \wedge \beta \to \alpha$ and $(\beta \wedge (\beta \to \alpha)) \to \alpha \wedge \beta$ are tautologies

$$\tau(\alpha) \geq \tau(\alpha \wedge \beta) \geq \tau(\beta \wedge (\beta \to \alpha)) \geq \tau(\alpha) + \tau(\beta) - 1.$$

Since $\tau(\alpha)$ is positive, we have also that $\tau(\alpha) \geq \tau(\beta \to \alpha) \odot \tau(\beta)$. ∎

13. THE CRISP PART OF THE CONSIDERED LOGICS

Recall that in classical logic the theories correspond with the filters and the complete theories with the ultrafilters of **B**. We say that a fuzzy logic in **B** is an *extension of the classical logic* provided that the related class of crisp theories coincides with the class of filters. Recall some definitions. A nonempty set F of formulas is called *upper* if

$$\alpha \in F \text{ and } \beta \geq \alpha \Rightarrow \beta \in F.$$

An upper set F is a *filter* provided that

$$\alpha \in F \text{ and } \beta \in F \Rightarrow \alpha \wedge \beta \in F.$$

The class of filters is a closure system and we denote by Fil the related closure operator. Then Fil(X) denotes the filter generated by X.

DEFINITION 13.1. We say that a set F of formulas is *consistent* if

$$\alpha \in F, \beta \in F \Rightarrow \alpha \wedge \beta \neq \mathbf{0}.$$

We say that F is a *class of probable formulas* if F is consistent and upper. If F satisfies also

$$-\alpha \notin F \Rightarrow \alpha \in F,$$

then we say that F is *a complete class of probable formulas* (the nomenclature is only slightly different from the one proposed in Walley and Fine [13]).

The name is justified by the fact that, given any probability p, the cuts $O(p, \lambda)$ and $C(p, \lambda)$ are complete classes of probable formulas for every $\lambda \geq \frac{1}{2}$ and for every $\lambda > \frac{1}{2}$, respectively. Obviously, a proper filter is a class of probable formulas and an ultrafilter is a complete class of probable formulas.

PROPOSITION 13.2. *The class of probable formulas together with the whole set* $\mathbf{B}$ *define a closure system. The related closure operator Prob is defined by*

$$\mathrm{Prob}(X) = \begin{cases} \mathbf{B} & \textit{if } X \textit{ is inconsistent} \\ \{\alpha \mid \exists \beta \in X,\ \beta \leq \alpha\} & \textit{if } X \textit{ is consistent}. \end{cases}$$

The following proposition shows that the logic of the super-additive measures is not an extension of the classical logic but of the qualitative logic of the probable formulas defined in [13].

PROPOSITION 13.3. *The logic of the super-additive measures is not an extension of the classical logic. Namely,*

(i) *a set X of formulas is consistent in* $\mathcal{S}_{sa}$ *iff it is consistent in accordance with Definition* 13.1;

(ii) *for every* $X \subseteq \mathbf{B}$

$$\mathcal{D}(X) = \mathrm{Prob}(X);$$

(iii) *the consistent crisp theories coincide with the classes of probable formulas*;

(iv) *the complete crisp theories coincide with the complete classes of probable formulas*.

Proof. (i) By Proposition 7.2 the characteristic function χ_X of X is consistent iff there is no pair α, β of disjoint formulas in X (otherwise $\chi_X(\alpha) + \chi_X(\beta) = 1 + 1 > 1$).

(ii) Formula (7.2) says that if X is consistent then $\mathcal{D}(X)(\mathbf{1}) = 1$ and, if $\alpha \neq 1$,

$$\mathcal{D}(X)(\alpha) = \mathrm{Sup}\{\chi_X(\beta) \mid \beta \leq \alpha\}.$$

Then, $\mathcal{D}(X)$ is the characteristic function of $\mathrm{Prob}(X)$.

(iii) This is an immediate consequence of (ii).

(iv) Let τ be a complete crisp theory and set $F = \{\alpha \in \mathbf{B} \mid \tau(\alpha) = 1\}$. Assume that $-\alpha \notin F$, i.e., $\tau(-\alpha) = 0$. Then since $\tau(\alpha) + \tau(-\alpha) = 1$, we have that $\tau(\alpha) = 1$ and therefore $\alpha \in F$. This proves that F is a complete class of probable formulas. Conversely, it is immediate that if τ is the characteristic function of a complete class F of probable formulas then τ is complete.

Finally, to prove that the sa-logic is not an extension of the classical logics observe that, given a finite set S, the class $\{X \in P(S) \mid \mathrm{Card}(X) > \mathrm{Card}(-X)\}$ is a complete class of probable formulas that is not a filter. ▮

PROPOSITION 13.4. *The logic of the necessities, the ul-logic, and the logic of the envelopes are extensions of the classical logic. Namely,*

(i) *a set X of formulas is consistent iff it satisfies the finite intersection property*

(ii) *for every $X \subseteq \mathbf{B}$*

$$D(X) = \mathrm{Fil}(X)$$

(iii) *the consistent crisp theories are the proper filters of* **B**

(iv) *the complete crisp theories are the ultrafilters.*

Proof. The proof is a matter of routine. ▮

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