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★Fuzzy logic. (English summary)

Mathematical tools for approximate reasoning.
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This is an enjoyable book on "mathematical tools for approximate reasoning", an important new area of formal logic initiated by L. Zadeh in the first half of the 1970s and later extensively studied by many theoretical linguists and logicians [see, e.g., V. Novák, I. G. Perfil′eva and J. Močkoř, *Mathematical principles of fuzzy logic*, Kluwer Acad. Publ., Boston, MA, 1999; MR 2001a:03068]. The book "does not have any pretension of completeness". It is mainly an exposition of the author's own ideas and recent research (some joint with L. Biacino, A. Di Nola, F. Ferrante, L. Scarpati, R. Tortora, and M. Ying).

The titles and highlights of the chapters are as follows: (1) Abstract logic in a lattice. Let $\mathcal{D}$ be a closure operator in a complete lattice L (a map such that $x \le y \Rightarrow \mathcal{D}(x) \le \mathcal{D}(y)$; $x \le \mathcal{D}(x)$; $\mathcal{D}(\mathcal{D}(x)) = \mathcal{D}(x)$), let τ be a fixed point of $\mathcal{D}$ (i.e., $\tau \in L$ such that $\mathcal{D}(\tau) = \tau$), and let $\mathcal{M}$ be a class of elements of L excluding the top element 1 of L. Generalizing A. Tarski's viewpoint (according to which a monotone logic is a set $\mathbb{F}$ of "well-formed" formulas together with a closure operator (deduction operator) in the lattice $\mathcal{P}(\mathbb{F})$ of all subsets of $\mathbb{F}$), the author of the present book calls the pair $(L, \mathcal{D})$ an abstract deduction system, $\mathcal{D}$ itself a deduction operator, the element τ a theory (saying it is consistent if $\tau \ne 1$), an element $x \in L$ (such that $\mathcal{D}(x) = \tau$) a system of axioms for τ, the class $\mathcal{M}$ an abstract semantics, and the elements in $\mathcal{M}$ models (writing $m \models x$ for "$m \in \mathcal{M}$ is a model of $x \in L$" if $x \le m$). He calls the triplet $(L, \mathcal{D}, \mathcal{M})$ an abstract logic provided that the "completeness theorem" holds, i.e., $\mathcal{D} = \mathrm{Lc}$, where Lc is a "logical consequence" operator defined by $\mathrm{Lc}(x) = \bigwedge\{m \in \mathcal{M} |\, m \models x\}$. The author also introduces the notions of logical compactness and continuity for $(L, \mathcal{D}, \mathcal{M})$ (it is continuous if, for every directed class $\mathcal{C}$ of subsets of L, $\mathcal{D}(\lim \mathcal{C}) = \lim \mathcal{D}(\mathcal{C})$).

(2) Abstract fuzzy logic. Let $\mathbb{F}$ (as above) be an abstract set (whose elements the author calls formulas), U a complete lattice, and $\mathcal{F}(\mathbb{F})$ the lattice of all fuzzy (U-valued) subsets of $\mathbb{F}$. The author calls an abstract fuzzy deduction system (abstract fuzzy semantics, abstract fuzzy logic) any abstract deduction system (abstract semantics, abstract logic) in $\mathcal{F}(\mathbb{F})$. (This virtually agrees with notions introduced by J. Pavelka [Z. Math. Logik Grundlag. Math. **25** (1979), no. 1, 45–52; MR 80j:03038a].)

(3) Extending an abstract crisp logic. Given a closure operator $\mathcal{D}$ in the lattice $\mathcal{P}(\mathbb{F})$, the author defines the canonical extension $\mathcal{D}^*: \mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{F})$ of $\mathcal{D}$, the closure operator defined by $\mathcal{D}^*(\nu)(\alpha) = \bigvee\{\lambda \in U | \alpha \in \mathcal{D}(C(\nu,\lambda))\}$, where $C(\nu,\lambda) = \{\alpha \in \mathbb{F} | \ \nu(\alpha) \geq \lambda\}$ (the closed λ-cut of $\nu \in \mathcal{F}(\mathbb{F})$), in order to extend any crisp (abstract) logic to a fuzzy (abstract) logic.

(4) Approximate reasoning. The author shows that it is possible to define the continuous deduction operator of a fuzzy logic by a suitable extension of the notions of inference rule and proof (in the main using the results of Pavelka [op. cit.]). He defines a fuzzy H-system (deduction system in "Hilbert style") to be a pair $(a, \mathbb{R})$ consisting of a fuzzy subset a (the fuzzy subset of logical axioms) of $\mathbb{F}$ and a set $\mathbb{R}$ of fuzzy inference rules (maps from a subset of $\mathbb{F} \times \cdots \times \mathbb{F}$ to $\mathbb{F}$ together with maps from $U \times \cdots \times U$ to U preserving arbitrary joins in each variable). He shows that the deduction operator $\mathcal{D}: \mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{F})$ associated with $(a, \mathbb{R})$ (drawing the best possible valuation from initial information) defined by $\mathcal{D}(\nu)(\alpha) = \bigvee\{\mathrm{Val}(\pi,\nu) | \ \pi \text{ is a proof of } \alpha\}$ (where $\mathrm{Val}(\pi,\nu)$ is the valuation of π with respect to ν) is a continuous fuzzy closure operator and that, for every continuous fuzzy closure operator $\mathcal{D}$, a fuzzy H-system exists whose associated deduction operator coincides with $\mathcal{D}$. The author calls a fuzzy Hilbert logic any pair $(\mathcal{S}, \mathcal{M})$, where $\mathcal{M}$ is a fuzzy semantics and $\mathcal{S}$ a fuzzy H-system whose deduction operator coincides with the logical consequence operator Lc associated with $\mathcal{M}$.

(5) Logic as management of constraints on the truth values. The author considers sets (of "formulas") with values in a lattice $\mathbb{C}$ of "constraints" (subsets of U) with respect to the "improvement" relation $\leq$ (by setting $X \leq Y \Leftrightarrow Y \subseteq X$). He introduces $\mathbb{C}$-inference rules, $\mathbb{C}$-fuzzy H-systems and $\mathbb{C}$-Hilbert logics (able to "manage" the constraints). To illustrate this, he considers Zadeh logic, Boolean logic, and probability logic. (6) Canonical extension of a crisp Hilbert logic. The author applies the results of Chapter 3 to H-systems. (7) Graded consequence relations. The author associates a chain $((\mathbb{F}, \mathcal{D}_\lambda))_{\lambda \in U}$ of deduction systems in $\mathcal{P}(\mathbb{F})$ with the "well-stratified" fuzzy deduction system $(\mathbb{F}, \mathcal{D})$ by setting, for every $\nu \in \mathcal{F}(\mathbb{F})$ and $\alpha \in \mathbb{F}$, $\mathcal{D}(\nu)(\alpha) = \bigvee\{\lambda \in U | \ \alpha \in \mathcal{D}_\lambda(C(\nu,\alpha))\}$ (see above). He shows that such systems one-to-one correspond with "graded consequence relations" (certain fuzzy relations g $(= g(X \vdash \alpha))$ from $\mathcal{P}(\mathbb{F})$ to $\mathbb{F}$ proposed in 1988 by M. K. Chakraborty) by $g(X \vdash \alpha) = \mathcal{D}(X)(\alpha)$.

(8) Truth-functional logic and fuzzy logic. Truth-functional multi-valued logic is examined. A strong connection between axiomatizability and continuity of the logical connectives is established. (9) Probabilistic fuzzy logics. The author proposes fuzzy logics, probabilistic in nature, which are strictly related to super-additive measures, upper-lower probabilities, lower envelopes and belief measures. (10) Fuzzy control and approximate reasoning. The author presents an approach to fuzzy control by translating any system of IF-THEN fuzzy rules into a system of fuzzy

clauses. (11) Effectiveness in fuzzy logics. The author extends to fuzzy sets the fundamental notions of decidability and recursive enumerability. On this basis he proposes some kind of "effectiveness" in a fuzzy logic by assuming that its deduction operator has a "computability" property (besides continuity), that the fuzzy subset of logical axioms is recursively enumerable and that the fuzzy rules are "computable". The author shows that the theory of effective abstract fuzzy deduction systems coincides with the theory of effective fuzzy H-systems. In conclusion, he obtains several interesting limitative results for fuzzy logic.

Reviewed by *R. Gylys*