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Sharpness Relation and Decidable Fuzzy Sets

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Abstract—**The concept of "decidability" for fuzzy sets is defined. We prove that there exist fuzzy sets which do not have a decidable "sharpened" version and fuzzy sets which are not "sharpened" versions of decidable fuzzy sets.**

I. Introduction

In this paper we define the decidability concept for fuzzy sets. We prove that a fuzzy set g exists, with $\{x/g(x)=1/2\}$ infinite, which does not have a decidable sharpened version. We also prove that a fuzzy set g exists, with $\{x/g(x)=1/2\}=\varnothing$, which is not a sharpened version of a decidable fuzzy set f with $\{x/f(x)=1/2\}$ finite or empty.

II. Notation and Mathematical Preliminaries

Let X be a set and L a totally ordered set. An *L-fuzzy set* [1],[3] is a map $f: X \to L$. We denote by $\mathfrak{L}(X, L)$ the class of L-fuzzy sets defined in X. If $f, g \in \mathfrak{L}(X, L)$, $f \wedge g$ and $f \vee g$ are defined point-by-point by $(f \wedge g)(x) = \inf\{f(x), g(x)\}$ and $(f \vee g)(x) = \sup\{f(x), g(x)\}$. If R and Q denote, respectively, the set of real numbers and the set of rational numbers, we set $[0,1] = \{x \in R/0 \leqslant x \leqslant 1\}$, $Q[0,1] = \{x \in Q/0 \leqslant x \leqslant 1\}$. If $x, y \in [0,1]$, we set $x \leqslant' y$ if $y \leqslant 1/2$ and $x \leqslant y$ or $y \geqslant 1/2$ and $x \geqslant y$. If $f, g \in \mathfrak{L}(X,[0,1])$, we set $f \leqslant' g$ if, for every $x \in X$, $f(x) \leqslant' g(x)$. In this case, we shall say that f is a *sharpened version* of g. If $f \in \mathfrak{L}$ and for every $x \in X$ it is $f(x) \in \{0,1\}$, we call f *crisp*.

Finally, if N is the set of natural numbers and $x \in N$, then φ_x is the partial recursive function of N from N with index x [2].

III. Decidable Fuzzy Sets

If X and Y are sets and $f: X \to Y$, we can pose the problem of the effective computability of f only if X and Y are *codified* by the elements of N. This happens, for example, if X is a finitely generated language and Y is the rational interval $Q[0,1]$.

Then, in order to define the decidable fuzzy sets, we must give the following definitions.

Definition 1: We call $\mathfrak{L}(X, L)$ *codified* if there exist $c_1: N \to X$ and $c_2: N \to L$ such that c_1 and c_2 are one-to-one.

Definition 2: If $\mathfrak{L}(X, L)$ is codified by c_1 and c_2, then we set, for every $z \in N$, $f_z = c_2 \varphi_z c_1^{-1}$. We call the fuzzy set f *decidable* if $f = f_z$ by a suitable $z \in N$.

By definition 2, if f_z is total, then f_z is a decidable fuzzy set.

Definition 3: Let $\mathfrak{L}(X, L)$ be codified by c_1 and c_2. Then we call $\mathfrak{L}(X, L)$ *computable* if L is a computable lattice; in other words, if the functions $f: N \times N \to N$ and $g: N \times N \to N$ defined by $f(m,n) = c_2^{-1}(c_2(n) \wedge c_2(m))$ and $g(m,n) = c_2^{-1}(c_2(n) \vee c_2(m))$ are computable.

The following propositions are immediate consequences, respectively, of the *s-m-n* theorem and Rice's theorem [2].

Proposition 1: Let $\mathfrak{L}(X, L)$ be computable. Then there exist two total recursive functions d and c such that, if f_x and f_y are decidable fuzzy sets, then $f_{d(x,y)} = f_x \wedge f_y$ and $f_{c(x,y)} = f_x \vee f_y$.

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Note that from Proposition 1 it follows that, if $\mathfrak{L}(X, L)$ is computable, then the decidable fuzzy set forms a sublattice of $\mathfrak{L}(X, L)$.

Proposition 2: There is not a uniform effective method to decide if $f_x = f_y$, $f_x \leqslant f_y$, or if f_x is crisp.

IV. Sharpened Version and Decidability

In order to avoid formal complications, we set $X = N$. Moreover, we examine only the particular case $\mathfrak{L} = \mathfrak{L}(X, Q[0,1])$. Obviously, $\mathfrak{L}$ is computable. It is easy to prove that a function $cr(x)$ exists such that, if f_x is a decidable fuzzy set, then $f_{cr(x)}$ is crisp and $f_{cr(x)} \leqslant' f_z$. In other words, every decidable fuzzy set f can be sharpened to a decidable crisp fuzzy set. This is not always possible if f is not decidable. In order to prove this, we give the following proposition.

Proposition 3: If, for every $a \in Q[0,1]$, we set

$$g_a(x) = \begin{cases} 1 - f_x(x) & \text{if } f_x(x) \text{ is convergent and } f_x(x) \neq 1/2 \\ a & \text{otherwise,} \end{cases}$$

then: a) a total recursive function h exists such that, for every decidable fuzzy set f_z, $f_z(h(z)) \not\leqslant' g_a(h(z))$; b) for every decidable fuzzy set f_z, the set $\{x/f_z(x) \not\leqslant' g_a(x)\}$ is infinite; and c) for every decidable fuzzy set f_z, if $g_a \leqslant' f_z$, then $\{x/f_z(x) = 1/2\}$ is infinite.

Proof: In order to prove a), let h be an injective total recursive function such that

$$f_{h(z)}(x) = \begin{cases} 0 & \text{if } f_z(x) \text{ is convergent and } f_z(x) \leqslant 1/2 \\ 1 & \text{if } f_z(x) \text{ is convergent and } f_z(x) > 1/2 \\ \text{divergent if } f_z(x) \text{ is divergent.} \end{cases}$$

By hypothesis, f_z is total; hence, $f_{h(z)}$ is total and $g_a(h(z)) = 1 - f_{h(z)}(h(z)) \in \{0,1\}$ for every $z \in N$. If $g_a(h(z)) = 1$, then $f_{h(z)}(h(z)) = 0$ and $f_z(h(z)) \leqslant 1/2$. If $g_a(h(z)) = 0$, then $f_{h(z)}(h(z)) = 1$ and $f_z(h(z)) > 1/2$. Hence, it is always $f_z(h(z)) \not\leqslant' g_a(h(z))$.

In order to prove b), we recall that the set $X_z = \{i/f_i = f_z\} = \{i/\varphi_i = \varphi_z\}$ is infinite. Moreover, for every $i \in X_z$, $f_z(h(i)) = f_i(h(i)) \not\leqslant' g_a(h(j))$. Since h is injective, b) is proved.

In order to prove c), let f_z be a decidable fuzzy set such that $g_a \leqslant' f_z$. Then also $g_a(z) \leqslant' f_z(z)$. If we suppose that $f_z(z) \neq 1/2$, then $g_a(z) = 1 - f_z(z)$, and therefore $1 - f_z(z) \leqslant' f_z(z)$. If $f_z(z) < 1/2$, then $f_z(z) \geqslant 1 - f_z(z)$, and hence $1/2 \leqslant f_z(z)$, while $f_z(z) > 1/2$ implies that $1 - f_z(z) \geqslant f_z(z)$, and hence $1/2 \geqslant f_z(z)$ is absurd. Then it is $f_z(z) = 1/2$ and, for every $i \in X_z$, $f_i(i) = 1/2$. It follows that, for every $i \in X_z$, $f_z(i) = f_i(i) = 1/2$. In conclusion, the set $\{x/f_z(x) = 1/2\}$ is infinite.

Corollary: A fuzzy set g exists, with $\{x/g(x) = 1/2\}$ infinite, which does not have a decidable sharpened version. Moreover, a fuzzy set g exists, with $\{x/g(x) = 1/2\} = \varnothing$, which is not a sharpened version of a decidable fuzzy set f with $\{x/f(x) = 1/2\}$ finite or empty.

Proof: It suffices to set $f = g_a$ with $a = 1/2$ and, respectively, $a \neq 1/2$.

Note that the fuzzy sets g_a have interesting properties. For example, $a \leqslant' b$ implies that $g_a \leqslant' g_b$. Moreover, a partial recursive function h exists such that, for every $y \in Q[0,1]$, if $y \neq 1/2$, then $g_a(h(y)) = y$. To prove this, it is sufficient to suppose that h is a function such that $f_{h(y)}$ is the total recursive function constantly equal to $1 - y$. Then $g_a(h(y)) = 1 - f_{h(y)}(h(y)) = y$. It is also easy to prove that the set $\{x/g_a(x) = y\}$ is infinite.

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