GIANGIACOMO GERLA AND VIRGINIA VACCARO

Modal Logic and Model Theory

Abstract. We propose a first order modal logic, the ***QS4E***-logic, obtained by adding to the well-known first order modal logic ***QS4*** a *rigidity axiom schemas*: $A \to \Box A$, where A denotes a basic formula. In this logic, the *possibility* entails the possibility of extending a given classical first order model. This allows us to express some important concepts of classical model theory, such as existential completeness and the state of being infinitely generic, that are not expressibile in classical first order logic. Since they can be expressed in $L_{\omega_1\omega}$-logic, we are also induced to compare the expressive powers of ***QS4E*** and $L_{\omega_1\omega}$. Some questions concerning the power of rigidity axiom are also examined.

1. Introduction

This work represents an attempt to establish a "bridge" between first order modal logic and classical model theory. To this purpose, we define a new modal system, the ***QS4E***-logic, obtained by adding to the first order modal logic ***QS4*** a "rigidity axiom" schema (see also [3] and [4])

(E) $\quad A \to \Box A$

where A is any basic formula. We prove completeness and compactness theorems for ***QS4E***. In the semantics which we propose for ***QS4E*** a formula of the type $\Diamond A$ is interpreted as the possibility of extending a given model to another model in which A holds. This enables us to express some recognized concepts of model theory, such as being existentially complete (Prop. 3.1) and infinitely generic (Prop. 4.4) which are not expressible in classical first order logic (for example, see [6] Corollary 3.17 and Theorem 14.13). Then we can derive the well-known results about the inductiveness of the class of infinitely generic models and of the class of existentially complete models of a given theory (Prop. 5.4) from a more general proposition about ***QS4E*** (Prop. 5.3). Other examples of the possibility of proving results of classical model theory via modal logic, are expressed in Prop. 3.1 (equivalence between (ii) and (iii)) and in Prop. 4.4 (equivalence among (ii), (iii), (iv) and (v)). Since existential completeness and being infinitely generic are concepts which are axiomatizable also in $L_{\omega_1\omega}$ (see [6] pag. 111), we compare the expressive powers of ***QS4E*** and $L_{\omega_1\omega}$.

Finally, reversing the above point of view, we utilize the stated relationship between modal logic and classical model theory in order to prove some results concerning the rigidity axiom.

2. The $QS4E$-Logic

In the sequel L denotes a classical first order language with equality and ML its modal extension. Namely, we assume $\neg$, $\wedge$, $\vee$, $\exists$, $\Box$ to be primitive connectives of ML and $\rightarrow$, $\leftrightarrow$, $\forall$, $\Diamond$ abbreviations. $QS4E$ is used for the logic obtained by adding to the quantified modal logic with equality $QS4$ the axiom schemata

(E) $\quad A \rightarrow \Box A$

where A stands for a *basic* formula, that is an atomic formula or the negation of an atomic formula of L.

Recall that the following formulas are axioms (or theorems) of $QS4$ and hence of $QS4E$.

(1) $\quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(2) $\quad \Box A \rightarrow A$

(3) $\quad \Box A \rightarrow \Box\Box A$

(4) $\quad \exists x_1 \ldots x_n \Box A \rightarrow \Box(\exists x_1 \ldots x_n A)$

(5) $\quad \Box A \vee \Box B \rightarrow \Box(A \vee B)$

(6) $\quad \Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$

Observe also that in $QS4$ the following formulas are consequences of (E).

(7) $\quad x = y \leftrightarrow \Box(x = y)$

(8) $\quad x \neq y \leftrightarrow \Box(x \neq y)$.

For any ML-formula A we define $\vdash_{QS4E} A$ as usual. If Γ is a set of formulas of ML, then $\Gamma \vdash_{QS4E} A$ means that $\vdash_{QS4E} B_1 \wedge \ldots \wedge B_n \rightarrow A$ for suitable $B_1, \ldots, B_n \in \Gamma$. Sometimes we shall write $\vdash A$ and $\Gamma \vdash A$ instead of $\vdash_{QS4E} A$ and $\Gamma \vdash_{QS4E} A$, respectively. Such a definition of "deduction from hypotheses" entails that, even if the necessity rule is assumed, $A \vdash \Box A$ does not hold, in general.

An adequate semantics for $QS4E$ is obtained by the following definitions. A $QS4$-*frame* is a pair $(W, \leqslant)$ where W is a class and $\leqslant$ is a reflexive and transitive relation on W. A $QS4E$-*modal structure* $S = (\mathcal{M}, W, \leqslant)$ for ML consists of a $QS4$-frame $(W, \leqslant)$ and a family $\mathcal{M} = (M_w)_{w \in W}$ of classical models $M_w = (D_w, \mathcal{I}_w)$ of L, where D_w is the *domain* and $\mathcal{I}_w$ the *interpretation* of M_w, such that $w \leqslant w'$ implies $M_w \subseteq M_{w'}$ (M_w is a submodel of $M_{w'}$) for every $w, w' \in W$. The interpretation of $=$ is the identity in every M_w. Once a single element w_0 has been selected in W, the pair (S, w_0) is called a $QS4E$-*model* for ML. Note that the proposed semantics is obtained by imposing conditions not only on the frame $(W, \leqslant)$, as it is usual for other modal logics, but also on the interpretations of L.

Every class Σ of classical model for L defines a $\boldsymbol{QS4E}$-modal structure for which $W = \Sigma$, $\mathcal{M}$ is the identity map and $\leqslant$ is $\subseteq$. We use the same symbol Σ to denote this $\boldsymbol{QS4E}$-modal structure. In particular, the class of models of a given first order theory defines a $\boldsymbol{QS4E}$-modal structure. In order to take account of such a type of $\boldsymbol{QS4E}$-modal structures we have extended the usual definition, assuming that W is a class and not necessarily a set.

If $w \in W$, $a_1, \ldots, a_p \in D_w$ and A is an ML-formula, then the relation $S, w \models A[a_1, \ldots, a_p]$ (*A is satisfied in S at w by $a_1, \ldots, a_p$*) is defined by recursion on the complexity of A, as usual (cf. for example [2]). The only interesting clause is when A has the form $\Box B$. In this case we set $S, w \models B[a_1, \ldots, a_p]$ if and only if, for every $w' \in W$ such that $w \leqslant w'$, we have $S, w' \models B[a_1, \ldots, a_p]$. We write $S, w \models A$ if $S, w \models A[a_1, \ldots, a_p]$ for every sequence $a_1, \ldots, a_p$ of elements of M_w, $S \models A$, *A is valid in S*, if $S, w \models A$ for every $w \in W$ and $\models_{QS4E} A$ (or simply $\models A$), *A is $\boldsymbol{QS4E}$-valid*, if $S \models A$ for every $\boldsymbol{QS4E}$-modal structure S.

If Γ is a set of formulas of ML, a $\boldsymbol{QS4E}$-model (S, w_0) is a *$\boldsymbol{QS4E}$-model of Γ* if $S, w_0 \models A$ for every $A \in \Gamma$. We write $\Gamma \models_{QS4E} A$ (or simply $\Gamma \models A$), and *A is a logical consequence of Γ*, if every $\boldsymbol{QS4E}$-model of Γ is a $\boldsymbol{QS4E}$--model of A.

It is easy to prove, by induction on the complexity of A, that the following weak forms of substitution of equivalents do hold.

(S1) If A' is an ML-formula obtained from A by replacing some occurrences of a subformula B of A with a formula B', we get:
$S \models B \leftrightarrow B'$ implies $S \models A \leftrightarrow A'$.

(S2) If B does not fall in the range of $\Box$, i.e. does not occur in a subformula of A of the form $\Box C$, then:
$\models (B \leftrightarrow B') \to (A \leftrightarrow A')$.

3. Completeness and compactness

Now, we shall prove a strong completeness theorem and a compactness theorem for $\boldsymbol{QS4E}$.

PROPOSITION 3.1 (STRONG COMPLETENESS THEOREM). *Let Γ be a set of ML-formulas, consistent with respect to $\boldsymbol{QS4E}$. Then there exists a $\boldsymbol{QS4E}$--model of Γ.*

PROOF. The strong completeness for $\boldsymbol{QS4E}$ can be proved by pointing out the strong completeness result for $\boldsymbol{QS4}$ already known (see, for example [5]). To this purpose, observe that if Γ is consistent with respect to $\boldsymbol{QS4E}$, then $\Gamma \cup E'$, where

$$E' = \{\Box(\forall x_1 \ldots x_n(A \to \Box A))/A \text{ any basic formula}\}$$

is consistent with respect to $\boldsymbol{QS4}$. Indeed, if we can derive a contradiction

C from $\Gamma \cup E'$ in $QS4$, then $\vdash_{QS4} A_1 \wedge \ldots \wedge A_p \wedge B_1 \wedge \ldots \wedge B_q \to C$ where $A_1, \ldots, A_p$ and $B_1, \ldots, B_q$ are suitable formulas of E' and Γ, respectively. Since $QS4E$ is an over-logic of $QS4$, we have also $\vdash_{QS4E} A_1 \wedge \ldots \wedge A_p \wedge \wedge B_1 \wedge \ldots \wedge B_q \to C$. Now, by Necessity Rule and Axiom Schema (E), we obtain $\vdash_{QS4E} A_i$ for $i = 1, \ldots, p$. Then by Modus Ponens we get $\vdash_{QS4E} B_1 \wedge \wedge \ldots \wedge B_q \to C$ and therefore $\vdash_{QS4E} C$, a contradiction.

From the consistency of $\Gamma \cup E'$ and the strong completeness theorem for $QS4$, it follows that there exists a $QS4$-model (S, w_0), with $S = (\mathcal{M}, W, \leqslant)$, where W is a set, $w_0 \in W$, $\leqslant$ is a reflexive and transitive relation on W, $\mathcal{M} = (M_w)_{w \in W}$ is a family of models, $M_w = (D_w, \mathcal{I}_w)$ such that $=$ is interpreted as identity and $D_w \subseteq D_{w'}$ for every $w, w' \in W$, $w \leqslant w'$. Moreover, as we know, it is not restrictive to assume that $w_0 \leqslant w$ for every $w \in W$. We shall prove that, for every $w, w' \in W$, if $w \leqslant w'$ then M_w is a submodel of $M_{w'}$. Now, let B be a basic formula, then, by hypothesis, $S, w_0 \models \Box \forall x_1 \ldots x_n (B \to \Box B)$. In particular, if A is atomic and n-ary, then $S, w \models \forall x_1 \ldots x_n (A \to \Box A)$ and $S, w \models \forall x_1 \ldots x_n (\neg A \to \Box(\neg A))$. So, if $a_1, \ldots, a_n$ are elements of D_w such that $(a_1, \ldots, a_n) \in \mathcal{I}_w(A)$, whence $S, w \models A[a_1, \ldots, a_n]$, then $S, w' \models A[a_1, \ldots, a_n]$ and therefore $(a_1, \ldots, a_n) \in \mathcal{I}_{w'}(A)$. Conversely, if $(a_1, \ldots, a_n) \in \mathcal{I}_{w'}(A)$ then $S, w' \models A[a_1, \ldots, a_n]$. Since $S, w \models \neg A \to \Box(\neg A)$, we have $S, w \models A[a_1, \ldots, a_n]$ and therefore $(a_1, \ldots, a_n) \in \mathcal{I}_w(A)$. In conclusion, $\mathcal{I}_w(A) = \mathcal{I}_{w'}(A) \cap D_w^n$.

This proves that M_w is a submodel of $M_{w'}$ and hence (S, w_0) is a $QS4E$-model of Γ.

Observe that the $QS4E$-model given by Proposition 3.1 is constructed in such a way that W is a set and $\mathrm{card}[(\bigcup_{w \in W} D_w) \cup W] = \mathit{card}(L)$. Then, despite the fact that W is sometimes a proper class, we can work out Löwenheim-Skolem type theorems for the $QS4E$-logic.

PROPOSITION 3.2. *If Γ is a set of ML-formulas, and A is any ML-formula, then $\Gamma \models_{QS4E} A$ if and only if $\Gamma \vdash_{QS4E} A$.*

PROOF. If Γ is inconsistent, the proof is obvious. Otherwise, assume $\Gamma \models A$ and Γ *non* $\vdash A$: then $\Gamma \cup \neg A$ is consistent. By Proposition 3.1, there exists a $QS4E$-model of $\Gamma \cup \neg A$. This contradicts the hypothesis $\Gamma \models A$.

Conversely, it is a matter of routine to prove that every axiom of $QS4E$ is $QS4E$-valid. Moreover, the rules of inference (in particular the necessity rule), preserve $QS4E$-validity. This proves that $\vdash A$ implies $\models A$.

Now, if $\Gamma \vdash A$, then $\vdash A_1 \wedge \ldots \wedge A_n \to A$ for suitable $A_i \in \Gamma$ and for that reason $\models A_1 \wedge \ldots \wedge A_n \to A$. Now, if (S, w_0) is any $QS4E$-model of Γ, then (S, w_0) is a $QS4E$-model of $\{A_1, \ldots, A_n\} \subseteq \Gamma$ and, therefore, of A.

Thus $\Gamma \models A$.

PROPOSITION 3.3 (COMPACTNESS THEOREM). *Let Γ be a set of ML-for-*

*mulas such that every finite subset of Γ has a **QS4E**-model.* Then Γ has a **QS4E**-model.

PROOF. It follows from Proposition 3.1.

4. Existentially complete models and model completeness

Now we shall show that some well known classes of models, which are not definible in classical first order logic are, in a sense, definable in **QS4E**. The following definitions generalize a notion of classical model theory. A **QS4E**-model (S, w_0) of ML is *existentially complete* if, for every $w \in W$ such that $w_0 \leqslant w$, M_{w_0} is existentially complete in M_w [6].

If Σ is a class of classical structures for L and $M \in \Sigma$ it is easy to prove that the **QS4E**-model (Σ, M) is existentially complete with regard to the above definition if and only if M is existentially complete in Σ with respect to the classical definition.

PROPOSITION 4.1. *The following are equivalent:*

(i) (S, w_0) *is existentially complete;*

(ii) $S, w_0 \models \Diamond A \leftrightarrow A$ *for every existential L-formula* A;

(iii) $S, w_0 \models \Diamond A \leftrightarrow A$ *for every universal-existential L-formula* A.

PROOF. (i) $\Rightarrow$ (ii). Let $a_1, \ldots a_p$ be elements of M_{w_0}, then $S, w_0 \models \Diamond A[a_1, \ldots, a_p]$ if and only if there exists $w \in W$ such that $w_0 \leqslant w$ and $S, w \models A[a_1, \ldots, a_p]$. Since A is a formula of L, this is equivalent to say that $M_w \models A[a_1, \ldots, a_p]$. But M_{w_0} is existentially complete in M_w, so $M_{w_0} \models A[a_1, \ldots, a_p]$ and therefore $S, w_0 \models A[a_1, \ldots, a_p]$. In conclusion $S, w_0 \models \Diamond A \rightarrow A$, and thus $S, w_0 \models \Diamond A \leftrightarrow A$.

(ii) $\Rightarrow$ (iii). Let A be the universal-existential L-formula $\forall x_1 \ldots \ldots x_n \exists y_1 \ldots y_m B$, with B quantifier-free. Then by (4) $\models \Diamond \forall x_1 \ldots x_n \exists y_1 \ldots \ldots y_m B \rightarrow \forall x_1 \ldots x_n \Diamond \exists y_1 \ldots y_m B$. Now, by hypothesis, $S, w_0 \models \Diamond \exists y_1 \ldots \ldots y_m B \leftrightarrow \exists y_1 \ldots y_m B$, so by S2 $S, w_0 \models \Diamond \forall x_1 \ldots x_n \exists y_1 \ldots y_m B \rightarrow \forall x_1 \ldots \ldots x_n \exists y_1 \ldots y_m B$. The converse of (2) yields $S, w_0 \models \forall x_1 \ldots x_n \exists y_1 \ldots y_m B \leftrightarrow \Diamond \forall x_1 \ldots \ldots x_n \exists y_1 \ldots y_m B$, and (iii) is proved.

(iii) $\Rightarrow$ (i). Immediate.

From the above Proposition, it follows, for example, that if Σ is the class of all fields then a field F is algebraically closed if and only if (Σ, F) is a **QS4E**-model of $\Gamma = \{A \leftrightarrow \Diamond A / A$ existential L-formula$\}$, where L is the language of the theory of fields. In general, if Σ is a class of structures of the same type, then we can assume Γ as a system of axioms in **QS4E** for the existentially complete elements of Σ.

Now we want to express the notion of model-completeness. A **QS4E**-modal structure S is *model complete* if, for every $w, w' \in W$ such that $w \leqslant w'$, $M_{w'}$ is an elementary extension of M_w. Observe that the concept of model-completeness is relative to **QS4E**-modal structures while existential completeness to **QS4E**-models. A class Σ of models of L is model complete in the sense of the classical definition [6] if and only if the associated *QS4E*-modal structure is model complete according to the above definition.

PROPOSITION 4.2. *Let S be a QS4E-modal structure, then the following are equivalent:*

(i) *S is model complete;*
(ii) *$S \models A \leftrightarrow \Box A$ for every L-formula A;*
(iii) *$S \models A \leftrightarrow \Box A$ for every ML-formula A.*

PROOF. (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iii). We proceed by induction on the number n of occurrences of $\Box$ in A. If $n = 0$ then A is a formula of L and the assertion holds by hypothesis. If $n \geqslant 1$, let $\Box B$ be a subformula of A such that B is an L-formula. From (ii) it follows that $S \models B \leftrightarrow \Box B$. Then, if A' is that formula obtained from A by substituting B for $\Box B$ we have, by inductive hypothesis, $S \models A' \leftrightarrow \Box A'$ and, by S1, $S \models A \leftrightarrow A'$. Still, by S1, it follows $S \models A \leftrightarrow \Box A$.

(iii) $\Rightarrow$ (i). Immediate.

In other words, Proposition 4.2 says that S is model complete if and only if the "theory" $\{A/S \models A\}$ "collapses", i.e. every ML-formula is equivalent to the L-formula obtained from it by deleting every occurrence of $\Box$.

5. Infinite forcing

Let S be a modal structure, $A(x_1, \ldots, x_p)$ a formula of L whose free or bound variables are among $x_1, \ldots, x_p$, $w \in W$ and $a_1, \ldots, a_p$ elements of M_w. So, the relation $S, w \Vdash A[a_1, \ldots, a_p]$ ((S, w) *infinitely forces* A *in* $a_1, \ldots, a_p$) is defined inductively as the satisfiability relation $S, w \models A[a_1, \ldots, a_p]$ except for what concerns the negation, for which we have:

$S, w \Vdash \neg B[a_1, \ldots, a_p]$ if and only if for all $w' \in W$, such that $w \leqslant w'$, S, w *non* $\Vdash B[a_1, \ldots, a_p]$.

A *QS4E*-model (S, w_0) is *infinitely generic* if for every formula $A(x_1, \ldots, x_p)$ of L and $a_1, \ldots, a_p$ elements of M_{w_0}, either $S, w_0 \Vdash A[a_1, \ldots, a_p]$ or $S, w_0 \Vdash A[a_1, \ldots, a_p]$.

The notions of infinite forcing and infinitely generic coincide with the classical ones if S is the modal structure associated to a class of models of L [6].

Now, in order to express the infinite forcing in the *QS4E*-logic, we introduce a suitable translation τ from L-formulas into ML-formulas.

DEFINITION 5.1. The map τ from *L-formulas into ML-formulas is* defined by recursion on the complexity of L-formulas by setting:

(i) if A is atomic, then $\tau A = A$;
(ii) if $A = B \vee C$, then $\tau(B \vee C) = \tau B \vee \tau C$;
(iii) if $A = B \wedge C$, then $\tau(B \wedge C) = \tau B \wedge \tau C$;
(iv) if $A = \exists x_h B(x_h)$, then $\tau(\exists x_h B(x_h)) = \exists x_h \tau B(x_h)$;
(v) if $A = \neg B$, then $\tau(\neg B) = \Box \neg \tau B$.

Obviously τ is not compatible with the equivalence, i.e. we cannot infer from $A \leftrightarrow B$ that $\tau(A) \leftrightarrow \tau(B)$. The translation τ is strictly related to the choice of $\wedge$, $\vee$, $\neg$, $\exists$, as primitive connectives of L. We made this a choice in order to be able to describe the infinite forcing as defined, for example, in [6]. Observe that translations into a modal language are already known in literature, see [7].

The following proposition shows the relation between $\models$ and $\Vdash$.

Proposition 5.2. *$S, w_0 \Vdash A[a_1, \ldots, a_p]$ if and only if $S, w_0 \models (\tau A)$ $[a_1, \ldots, a_p]$.*

Proof. The proof is by induction on the complexity of A.

The following proposition gives us two useful properties of τ.

Proposition 5.3. *The following hold:*

(i) $\vdash \tau(\neg A) \to \neg \tau A$;
(ii) $\vdash \tau A \to \Box \tau A$.

Proof. (i). It follows from (v) of Definition 5.1. and (2).

(ii). The proof is by induction on the complexity of A. If A is atomic, then $\tau A = A$ and (ii) follows from axiom schema (E).

Let $A = B \vee C$, then by inductive hypothesis, $\vdash \tau(B) \to \Box(\tau(B))$ and $\vdash \tau(C) \to \Box(\tau(C))$. Thus, $\vdash \tau(B) \vee \tau(C) \to \Box(\tau(B)) \vee \Box(\tau(C))$. From (5) we also have $\vdash \tau(B) \vee \tau(C) \to \Box(\tau(B) \vee \tau(C))$ and therefore $\vdash \tau(B \vee C) \to \Box(\tau(B \vee C))$.

If $A = B \wedge C$, by inductive hypothesis, we get $\vdash \tau(B) \wedge \tau(C) \to \Box(\tau(B)) \wedge \wedge \Box(\tau(C))$ and, by (6), $\vdash \tau(B) \wedge \tau(C) \to \Box(\tau(B) \wedge \tau(C))$, i.e. $\vdash \tau(B \wedge C) \to \Box \tau(B \wedge C)$.

If $A = \exists x_h B(x_h)$, then, by inductive hypothesis, $\vdash \tau(B) \to \Box(\tau(B))$ and hence $\vdash \exists x_h \tau(B) \to \exists x_h \Box(\tau(B))$. From (4), it follows that $\vdash \exists x_h \tau(B) \to \Box(\exists x_h \tau(B))$ and therefore $\vdash \tau(\exists x_h B) \to \Box \tau(\exists x_h B)$.

Finally, if $A = \neg B$, we must prove that $\vdash \Box(\neg(\tau B)) \to \Box(\Box(\neg(\tau B)))$, but this follows from (3), so (ii) is proved.

Proposition 5.4. *The following are equivalent:*

(i) *(S, w_0) is infinitely generic;*
(ii) *$S, w_0 \models \tau(A \vee \neg A)$ for every L-formula A;*
(iii) *$S, w_0 \models \Diamond(\tau A) \leftrightarrow \tau A$ for every L-formula A;*

(iv) $S, w_0 \vDash \neg(\tau A) \leftrightarrow \tau(\neg A)$ *for every L-formula A*;
(v) $S, w_0 \vDash A \leftrightarrow \tau(A)$ *for every L-formula A*.

PROOF. (i)$\Leftrightarrow$(ii). It follows from the definition of infinitely generic ***QS4E***-model, condition (ii) of Definition 5.1 and Proposition 5.2.

(ii)$\Rightarrow$(iii). From $S, w_0 \vDash \tau(A \vee \neg A)$ it follows $S, w_0 \vDash \tau A \vee \tau(\neg A)$ and, thus, $S, w_0 \vDash \tau A \vee \Box \neg \tau A$. This proves that $S, w_0 \vDash \neg \Box (\neg \tau A) \rightarrow \tau A$ and therefore, $S, w_0 \vDash \Diamond (\tau A) \rightarrow \tau A$.

(iii)$\Rightarrow$(iv). It suffices to observe that by (iii) $S, w_0 \vDash \Box(\neg \tau A) \leftrightarrow \neg \tau A$.

(iv)$\Rightarrow$(v). We proceed by induction on the complexity of A.

If A is atomic, (v) is obvious.

If $A = B \vee C$, $A = B \wedge C$ or $A = \exists x_h B(x_h)$, the inductive step follows from the definition of τ.

If $A = \neg B$, then $S, w_0 \vDash \neg B$ if and only if S, w_0 *non*$\vDash B$ if and only if, by inductive hypothesis, S, w_0 *non* $\vDash \tau(B)$. Then, (iv), yields $S, w_0 \vDash \neg B$ if and only if $S, w_0 \vDash \tau(\neg B)$.

(v)$\Rightarrow$(i). Since $S, w_0 \vDash A$ or $S, w_0 \vDash \neg A$ and, by hypothesis, $S, w_0 \vDash A \leftrightarrow \tau A$ and $S, w_0 \vDash \neg A \leftrightarrow \tau(\neg A)$, from $S, w_0 \vDash A \vee \neg A$ it follows $S, w_0 \vDash \tau A \vee \tau(\neg A)$. Condition (i) follows from the definition of infinitely gener ic ***QS4E***-model and Proposition 5.2.

Proposition 5.4 shows that the infinitely generic ***QS4E***-models are the models of suitable axiom systems, for example, $\Gamma = \{\tau(A \vee \neg A)/A$ any L-formula$\}$. In particular, a classical structure M is infinitely generic in a class Σ of models of L if and only if (Σ, M) is a ***QS4E***-model of Γ. In this sense, we regard Γ as an axiom system for the infinitely generic structures.

6. Inductive classes

In order to give an example of an application of ***QS4E***-logic to classical model theory, we will derive from a general result on ***QS4E***-logic a well-known result in classical model theory. Namely, the existentially complete and infinitely generic structures of a given class of structures constitute inductive classes.

To this aim some definitions are introduced that generalize those given in literature. An *ascending chain* of models of L is a sequence $(M_n)_{n \in N}$ of models of L such that if $n \leqslant m$ then $M_n \subseteq M_m$. For the definition of the union, $\bigcup_{n \in N} M_n$, of an ascending chain see [2].

DEFINITION 6.1. A subclass Σ' of a class Σ of models of L is called *inductive in* Σ if for every ascending chain $(M_n)_{n \in N}$ of elements of Σ', from $\bigcup_{n \in N} M_n \in \Sigma$ it follows that $\bigcup_{n \in N} M_n \in \Sigma'$. If Σ' is inductive in the class of all models of L then we say that Σ' is *inductive*.

DEFINITION 6.2. An ML-formula A is *inductive in a class* Σ of models of L if $\Sigma' = \{M / \Sigma, M \models A\}$ is inductive in Σ. A is *inductive* if it is inductive in every class Σ.

PROPOSITION 6.3. *Let A be an ML-formula, then $\Box A$ is inductive. Besides, if A is inductive, $\Diamond A \leftrightarrow A$ is inductive.*

PROOF. The formula $\Box A$ is inductive, by means of simple considerations. Moreover, let Σ be any class of models of L, A an inductive formula, $\Sigma' = \{M \in \Sigma / \Sigma, M \models \Diamond A \leftrightarrow A\}$ and $(M_n)_{n \in N}$ an ascending chain of elements of Σ' such that $\bar{M} = \bigcup_{n \in N} M_n \in \Sigma$. We shall prove that $\bar{M} \in \Sigma'$, that is $\Sigma, \bar{M} \models \Diamond A \leftrightarrow A$. Assume that $\Sigma, \bar{M} \models \Diamond A$. Since every M_n is a submodel of $\bar{M}$, we have $\Sigma, M_n \models \Diamond A$ for every $n \in N$. As, by hypothesis, $\Sigma, M_n \models \Diamond A \leftrightarrow A$, we also get $\Sigma, M_n \models A$ for every $n \in N$. A being inductive, it follows that $\Sigma, \bar{M} \models A$. In conclusion, $\Sigma, \bar{M} \models \Diamond A \rightarrow A$ and therefore $\Sigma, \bar{M} \models \Diamond A \leftrightarrow A$.

PROPOSITION 6.4. *Let Σ be a class of models of L and $\mathscr{E}(\Sigma)$ and $\mathscr{I}(\Sigma)$ the classes of models which are existentially complete and infinitely generic in Σ, respectively. Then, $\mathscr{E}(\Sigma)$ and $\mathscr{I}(\Sigma)$ are inductive in Σ.*

PROOF. $\mathscr{E}(\Sigma)$ is inductive by Proposition 4.1. (ii) and Proposition 6.3. From (2) and Proposition 5.3 (ii), it follows that $\vdash \tau A \leftrightarrow \Box \tau A$ for every ML-formula A. Then, from Proposition 5.4 (ii) and Proposition 6.3, it follows that $\mathscr{I}(\Sigma)$ is inductive.

7. *QS4E*-logic and $L_{\omega_1,\omega}$-Logic

In this section we want to compare the expressive powers of ***QS4E*** and $L_{\omega_1,\omega}$. This comparison raises some difficulties, since the models of ***QS4E*** and of $L_{\omega_1,\omega}$ are different "objects". In spite of this, we can ask whether there exists a translation f from the ML-formulas into the $L_{\omega_1,\omega}$-formulas such that $\Sigma, M \models A$ if and only if $M \models f(A)$ for any class Σ of models of L, any $M \in \Sigma$ and any formula A of ML. Conversely, we can ask whether there exists a similar translation from $L_{\omega_1,\omega}$-formulas into ML-formulas. The answers to both questions are negative. In order to prove this, we suppose in the sequel that L is the pure identity language. Then a model of L is any set, and a ***QS4E***-modal structure for ML is any family $(X_w)_{w \in W}$ of sets, with $(W, \leqslant)$ a ***QS4E***-frame, such that $X_w \subseteq X_{w'}$ for every $w, w' \in W$ such that $w \leqslant w'$. In particular every class Φ of sets defines a ***QS4E***-modal structure. The following proposition shows that, if Σ is the class of all sets, then, relatively to the ***QS4E***-modal structure Σ, we can eliminate the modal operators.

PROPOSITION 7.1. *Let L be the pure identity language and Σ the class of all models of L. Then for every ML-formula A there exists an L-formula A^* such that $\Sigma \models A \leftrightarrow A^*$.*

PROOF. First, we assume that $A = \Box B$ with B an L-formula. Now, from the theorem of Elimination of Quantifiers for the pure identity language (see [2], Theorem 1.5.7), B is equivalent to a formula of the type $(A_1 \vee B_1) \wedge \ldots \wedge (A_p \vee B_p)$, where the A_i's are L-formulas without quantifiers and the B_i's are sentences of L. Then, $\vdash \Box B \leftrightarrow \Box\big((A_1 \vee B_1) \wedge \ldots \wedge (A_p \vee B_p)\big)$ and by (6)

(9) $\quad \vdash \Box B \leftrightarrow \Box(A_1 \vee B_1) \wedge \ldots \wedge \Box(A_p \vee B_p).$

Moreover, from (5) it follows, for $i = 1, \ldots, p$,

(10) $\quad \vdash \Box A_i \vee \Box B_i \rightarrow \Box(A_i \vee B_i)$

and by (1) $\vdash \Box(\neg A_i \rightarrow B_i) \rightarrow (\Box \neg A_i \rightarrow \Box B_i)$, i.e.

(11) $\quad \vdash \Box(A_i \vee B_i) \rightarrow \Diamond A_i \vee \Box B_i.$

As each A_i is quantifier-free, from axiom schema (E) it follows that $\vdash A_i \leftrightarrow \Box A_i$ and $\vdash A_i \leftrightarrow \Diamond A_i$. Then, from (10) and (11) it follows that

(12) $\quad \vdash \Box(A_i \vee B_i) \leftrightarrow A_i \vee \Box B_i.$

Now, the B_i's are sentences of the pure identity language. These sentences are equivalent to the assertion that the cardinality of the model belongs either to an empty or to a finite set I of finite cardinals, or to the complement of such a set. In the former case $\Box B_i$ is always false in the $QS4E$-modal structure Σ, and

(13) $\quad \Sigma \models \Box B_i \leftrightarrow B_i \wedge \neg B_i.$

In the latter, $\Sigma, M \models \Box B_i$ if and only if for all $M' \supseteq M$, $card(M') \notin I$, if and only if $card(M) \geqslant max I$. If $m = max I$ and C_m denotes an L-formula expressing that there are more than m elements, then

(14) $\quad \Sigma \models \Box B_i \leftrightarrow C_m.$

In conclusion, from (9), (12), (13) and (14) the desired result, for the formula $\Box B$, follows.

Now, let A be any ML-formula. In this case one proceeds by induction on the number n of occurrences of $\Box$ in A. If $n = 0$, the assertion is obvious. If $n > 0$, there exists a subformula $\Box B$ of A with B an L-formula. Now, let C be an L-formula such that $\Sigma \models \Box B \leftrightarrow C$ and A' the formula obtained from A by substituting C for $\Box B$. Then, from S1, it follows that $\Sigma \models A \leftrightarrow A'$. Since the modal degree of A' is $n-1$, by inductive hypothesis, there exists an L-formula A^* such that $\Sigma \models A' \leftrightarrow A^*$ and hence $\Sigma \models A \leftrightarrow A^*$.

Observe that (13) and (14) hold only for the $QS4E$-modal structure Σ and that they can not be substituted by the stronger assertions $\vdash \Box B_i \leftrightarrow B_i \wedge \neg B_i$ and $\vdash \Box B_i \leftrightarrow C_m$. It follows that in Proposition 7.1 we are not alleged to substitute $\Sigma \models A \leftrightarrow A^*$ by $\vdash A \leftrightarrow A^*$. On the contrary, Proposition 6.3 shows that such a translation of $QS4E$ in the classical logic is impossible.

The following proposition proves that, in general, there is no reduction of $L_{\omega_1,\omega}$-logic to $QS4E$-logic.

PROPOSITION 7.2. *Let L and Σ be as in Proposition 7.1. Then there exists no translation f from the $L_{\omega_1,\omega}$-formulas into the ML-formulas such that*

$$M \models A \quad \textit{if and only if} \quad \Sigma, M \models f(A)$$

for every $L_{\omega_1,\omega}$-formula A and every $M \in \Sigma$.

PROOF. We proceed by absurd. Let A be a formula of $L_{\omega_1,\omega}$ such that $M \models A$ if and only if M is finite. Then, by hypothesis, Σ, $M \models f(A)$ if and only if M is finite. Now, from Proposition 7.1 it follows that there exists an L-formula $(f(A))^*$ such that $\Sigma, M \models f(A)$ if and only if $M \models (f(A))^*$ while it is well-known that finiteness is not definable in first order logic, a contradiction.

From Proposition 7.1, it follows that Σ, $M \models A$ if and only if $M \models A^*$, where $M \in \Sigma$ and A^* is an L-formula. Since A is also an $L_{\omega_1,\omega}$-formula, this proves that, relatively to Σ, a translation from $QS4E$ into $L_{\omega_1,\omega}$ does exist. In spite of that, the following proposition shows that there exists a $QS4E$-model which does not allow any translation from $QS4E$ into $L_{\omega_1,\omega}$. Then, in general, there is no translation of $QS4E$ in $L_{\omega_1,\omega}$ and these logics have incomparable expressive powers.

PROPOSITION 7.3. *Suppose $\Phi = \{X, Y, Z\}$ where X and Y are sets with n elements, $X \not\subseteq Z$ and $Y \subset Z$. Then there exists no translation f of ML-formulas into $L_{\omega_1,\omega}$-formulas such that*

$$\Phi, M \models A \quad \textit{if and only if} \quad M \models f(A)$$

for every ML-formula A and $M \in \Phi$.

PROOF. We proceed by absurd. Let A be an L-formula which expresses the existence of just n elements. Then Φ, $X \models \Box A$, and Φ, Y *non* $\models \Box A$ and hence, by hypothesis, $X \models f(\Box(A))$ and Y *non* $\models f(\Box(A))$. This is absurd. Indeed, X and Y have the same cardinality and therefore are isomorphic models of the pure identity language. This entails that X and Y verify the same $L_{\omega_1,\omega}$-formulas.

8. Overlogics of $QS4E$

An interesting question is to examine the overlogics of $QS4E$. We can obtain such overlogics either by adding new axioms or by imposing new conditions on modal structures. From the first point of view, it is natural to extend the "rigidity" axiom schema (E) to a larger class of formulas. We can extend (E) to every L-formula A quantifierfree. Let $\boldsymbol{QS4E}_1$ be the overlogic so obtained. The following proposition shows that $\boldsymbol{QS4E}_1$ coincides with $\boldsymbol{QS4E}$.

PROPOSITION 8.1. *$\boldsymbol{QS4E}_1$ coincides with $\boldsymbol{QS4E}$.*

PROOF. It suffices to prove that for every A quantifier-free we have: (a) $\vdash_{QS4E} A \leftrightarrow \Box A$, (b) $\vdash_{QS4E} A \leftrightarrow \Diamond A$. We proceed by induction on the complexity of A. Let A be an atomic formula, then by (E) $\vdash_{QS4E} A \leftrightarrow \Box A$. In order to obtain (b), we consider $\neg A$, then, by (E), $\vdash_{QS4E} \neg A \leftrightarrow \Box \neg A$. From this it follows that $\vdash_{QS4E} \Diamond A \leftrightarrow A$, i.e. (b). Let $A = B \vee C$, then by inductive hypothesis, $\vdash_{QS4E} B \leftrightarrow \Box B$ and $\vdash_{QS4E} C \leftrightarrow \Box C$, so $\vdash_{QS4E} B \vee C \leftrightarrow \Box B \vee \Box C$, but $\vdash_{QS4E} \Box B \vee \Box C \rightarrow \Box (B \vee C)$, then $\vdash_{QS4E} B \vee C \leftrightarrow \Box (B \vee C)$. In the same way we prove that $\vdash_{QS4E} B \vee C \leftrightarrow \Diamond (B \vee C)$. If $A = B \wedge C$, we proceed as above. If $A = \neg B$, then by inductive hypothesis, $\vdash_{QS4E} B \leftrightarrow \Box B$, so $\vdash_{QS4E} \Diamond \neg B \leftrightarrow \neg B$, that is $\vdash_{QS4E} A \leftrightarrow \Diamond A$, while, from the inductive hypothesis $\vdash_{QS4E} B \leftrightarrow \Diamond B$ it follows that $\vdash_{QS4E} A \leftrightarrow \Box A$.

Now, we extend (E) to every existential formula A of L, and denote this system with $\boldsymbol{QS4E}_2$

PROPOSITION 8.2. *$\boldsymbol{QS4E}_2$ coincides with $\boldsymbol{QS4E}$.*

PROOF. Let $A = \exists x_h B(x_h)$, with B a quantifier-free formula. From Proposition 8.1 it follows that $\vdash_{QS4E} B \leftrightarrow \Box B$ and, therefore, $\vdash_{QS4E} \exists x_h B(x_h) \leftrightarrow \exists x_h \Box B(x_h)$. Since $\vdash_{QS4E} \exists x_h \Box B(x_h) \rightarrow \Box \exists x_h B(x_h)$, we have $\vdash_{QS4E} \exists x_h B(x_h) \rightarrow \Box \exists x_h B(x_h)$. This proves that $\vdash_{QS4E} A \leftrightarrow \Box A$.

Now we use $\boldsymbol{QS4E}_3$ for the overlogic obtained by extending (E) to every universal L-formula A.

PROPOSITION 8.3. *The overlogic $\boldsymbol{QS4E}_3$ is a proper extension of $\boldsymbol{QS4E}$.*

PROOF. From Proposition 4.1 it follows that every model of $\boldsymbol{QS4E}_3$ is an existentially complete $\boldsymbol{QS4E}$-model. But $\boldsymbol{QS4E}$ has models that are not existentially complete. This proves that $\boldsymbol{QS4E}_3$ is a proper extension of $\boldsymbol{QS4E}$.

It is possible to consider many extensions of such a type, but we conclude with the case in which (E) is extended to every L-formula. We denote this overlogic with $\boldsymbol{QS4E}^*$

PROPOSITION 8.4. *The logic $\boldsymbol{QS4E}^*$ is a collapsing proper extension of $\boldsymbol{QS4E}$.*

PROOF. It suffices to use the equivalence between (ii) and (iii) of Proposition 4.2.

It is also interesting to consider the overlogics of ***QS4E*** which are obtained by adding some of the well-known modal formulas as new axioms. For example, let $\boldsymbol{QS4E}+\boldsymbol{BF}$ be the system in which the Barcan formula, BF, $\Diamond\exists xA\rightarrow\exists x\Diamond A$, is added as a new axiom.

PROPOSITION 8.5. *The logic* $\boldsymbol{QS4E}+\boldsymbol{BF}$ *is a collapsing extension of* ***QS4E***.

PROOF. We prove that, for every L-formula A, $\vdash_{QS4E+BF}A\leftrightarrow\Box A$ and $\vdash_{QS4E+BF}A\leftrightarrow\Diamond A$. We proceed by induction on the complexity of A. If A is atomic, or $A=B\vee C$, or $A=B\wedge C$, or $A=\neg B$, the proof is as in the Proposition 8.1. If $A=\exists x_h B(x_h)$, by inductive hypothesis, $\vdash_{QS4E+BF}B\rightarrow\Box B$ and then $\vdash_{QS4E+BF}\exists x_h B(x_h)\rightarrow\exists x_h\Box B(x_h)$. Since $\vdash_{QS4}\exists x_h\Box B(x_h)\rightarrow\Box\exists x_h B(x_h)$, we have $\vdash_{QS4E+BF}\exists x_h B(x_h)\rightarrow\Box\exists x_h B(x_h)$, that is $\vdash_{QS4E+BF}A\rightarrow\Box A$.

On the other hand, from BF and the inductive hypothesis $\vdash_{QS4E+BF}\Diamond B\rightarrow B$ it follows that $\vdash_{QS4E+BF}\exists x_h\Diamond B(x_h)\rightarrow\exists x_h B(x_h)$. Then $\vdash_{QS4E+BF}\Diamond\exists x_h B(x_h)\rightarrow\exists x_h B(x_h)$, that is $\vdash_{QS4E+BF}\Diamond A\leftrightarrow A$.

Since the Barcan formula is a theorem of ***QS5***, the Proposition 8.5 also proves that the ***QS5E*** system is a collapsing extension of ***QS4E***.

Finally, we observe that, from a semantical point of view, we can obtain overlogics of ***QS4E*** by defining $\models$ referring to particular subclasses of ***QS4E***-models. For example, we can investigate the overlogic obtained by considering only these ***QS4E***-modal structures determined by classes of classical models. It is an open interesting question to give a suitable set of axioms for this overlogic.

We can also consider only the ***QS4E***-modal structures S modal complete. In other words, we can substitute the condition: "$M_{w'}$ is an extension of M_w" by the stronger one: "$M_{w'}$ is an elementary extension of M_w", for every $w, w'\in W$ and $w\leqslant w'$. As it is proved in Proposition 3.2., the logic defined in that way collapses.

References

[1] K. A. BOWEN, ***Model Theory For Modal Logic,*** D. Reidel P. Company, London 1979.

[2] C. C. CHANG and H. J. KEISLER, ***Model Theory,*** North-Holland, Amsterdam 1973.

[3] K. FINE, *First-order modal theories I — Sets,* ***Nous,*** 15, (1981), pp. 117–206.

[4] K. FINE, *First-order modal theories III — Facts,* ***Synthese,*** 53, (1982), pp. 43–122.

[5] D. M. GABBAY, ***Investigations in Modal and Tense Logics with Applications to Problems in Philosophy and Linguistics,*** D. Reidel P. Company, Dordrecht-Holland/Boston-U. S. A., 1976.

[6] J. Hirschfeld and W. H. Wheeler, ***Forcing, Arithmetic, Division Rings***, Lecture Notes in Mathematics, vol. 454, Springer Verlag, 1975.
[7] J. C. C. McKinsey and A. Tarski, *Some theorems about the sentential calculi of Lewis and Heyting*, ***The Journal of Symbolic Logic***, vol. 13 (1948), pp. 1–15.

Institute of Mathematics
University of Napoli
Italy

Received November 15, 1982