DIFFERENTIAL GEOMETRY AND PDES

LUCA VITAGLIANO

Abstract. These are lecture notes of a 15 hour PhD course which I held at the University of Salerno in February, 2017.

Contents

1. Introduction

We often think of PDEs in terms of coordinates, but actually PDEs contain some information which is independent of the choice of coordinates. This information is the most relevant one, because it is unaffected by any arbitrary choice. The aim of this course is to develop a geometric language of PDEs which does only keep the relevant information: the one independent of the choice of coordinates. This is similar to what one does in algebraic geometry encoding a system of algebraic PDEs into a geometric object, in that case an algebraic variety.

2. Jet Spaces

2.1. **Multi-indexes.** Analitically, writing a (system) of (possibly non-linear) PDEs requires choosing some *independent variables* $x = (x^1, \ldots, x^n)$, some dependent variables $u = (u^1, \ldots, u^m)$, that should be understood as functions of x, and a set of equations

$$F_a\left(x, \ldots, u_I(x), \ldots\right) = 0$$

imposed on functions $u = u(x)$ and their derivatives $u_I = u_I(x) = \frac{\partial^{|I|} u}{\partial x^I}(x)$.

Remark 2.1 (Multiindex notation)**.** Let n be the number of independent variables. Put $I_n = \{1, \ldots, n\}$. Then, a lenght $k \geq 0$ multiindex is $I = (i_1, \ldots, i_k) \in I_n^{\times k}$. We also write $|I| := k$ (the *lenght* of I) and we identify two multi-indexes up to permutations of their entries. In other words

$$\{\text{multi-indexes}\} = \text{free commutative monoid generated by letters } \{1, \ldots, n\},$$

and for a multi-index I with entries $i_1, \ldots, i_k$, we write $I = i_1 \cdots i_k$. The lenght $I \mapsto |I|$ maps multi-indexes homomorphically to additive, non-negative integers. Finally, given a multi-index $I = (i_1, \ldots, i_k)$, we write

$$\frac{\partial^{|I|}}{\partial x^I} := \frac{\partial^k}{\partial x^{i_1} \cdots \partial x^{i_k}}$$

and it is well-defined. Notice that there are other possible notations.

Now on, if I is a multi-index, and f is a function of the x's, we also denote

$$f_{,I} := \frac{\partial^{|I|} f}{\partial x^I}$$

Exercise 2.2. Show that the (multiplicative) monoid of multi-indexes is isomorphic to the additive monoid $\mathbb{N}_0^n$. Describe an isomorphism and relate it to a different notational convention on multiple partial derivatives.

2.2. **Fiber bundles and their sections.** We want to define PDEs in a coordinate independent way. We will mainly deal with PDEs imposed on *sections of a fiber bundle*. Accordingly, the x's will be coordinates on a (base) manifold M and the u's fiber coordinates on a fiber bundle $E \to M$, so that $u = u(x)$ will be a section of E.

We now recall the relevant definitions. Let $M, \mathcal{F}$ be manifold, $\dim M = n$, $\dim \mathcal{F} = m$.

Definition 2.3. A rank m *fiber bundle* over M with *abstract fiber* $\mathcal{F}$ is a manifold E, $\dim E = n + m$ together with a smooth surjection $\pi : E \to M$ such that, for every point $x \in M$, there is an open neighborhood $U \ni x$ in M and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times F$ such that diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

commutes.

We adopt the following terminology:

- M is the *base manifold*.
- E is the *total space*.
- π is the *bundle projection*.

For $x \in M$, $\pi^{-1}(x)$, also denoted E_x, is an m-dimensional submanifold diffeomorphic to F and called *the fiber of E (growing) over x*.

U is a *trivializing neighborhood*, and ϕ_U is a *local trivialization*.

If $U = M$ we say that E is a *trivial bundle*: it's isomorphic to $M \times F$ with projection $M \times F \to M$, in a suitable sense.

Remark 2.4. Let $E \to M$ be a fiber bundle and $U \subseteq M$ a trivializing neighborhood. Shrinking U if necessary, we can assume that U is a coordinate neighborhood with coordinates $(x^1, \dots, x^n)$. Let V be a coordinate neighborhood on the fiber $\mathcal{F}$ with coordinates $(u^1, \dots, u^m)$. Then we get coordinates on $\phi_U^{-1}(U \times V)$ that we denote by $(x^1, \dots, x^n, u^1, \dots, u^m)$ again and call *bundle coordinates*.

We adopt the following terminology:

- $(x^1, \dots, x^n)$ are the *base coordinates* (they are constant along the fibers).
- $(u^1, \dots, u^m)$ are the *fiber coordinates* (they can be thought as coordinates along the fibers).

Definition 2.5. A (local) *section* of a bundle $E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. A *local section* is like a section but it is only defined on some open neighborhood. The *graph* of s is its image (it is an n-dimensional embedded submanifold, and $s : M \to s(M)$ is a diffeomorphism with inverse $\pi : s(M) \to M$).

Remark 2.6. A fiber bundle needs not to posses global sections, but it always possess local sections. Even more, for every $e \in E$ there is a local section s through e, i.e. $s(x) = e$, where $x = \pi(e)$.

In bundle coordinates a section is completely determined by functions

$$u^\alpha = s^\alpha(x) = s^*(u^\alpha), \quad \alpha = 1, \dots, m.$$

We write

$$s : u^\alpha = s^\alpha(x).$$

So *local sections of fiber bundles* are geometric (and coordinate independent) models for vector valued functions of n variables $x = (x^1, \dots, x^n)$.

Definition 2.7. Let $E, G \to M$ be fiber bundles over M. A *bundle map*, or a *bundle morphism* is a smooth map $F : E \to G$ such that diagram

$$\begin{tikzcd}
E \arrow[rr, "\Phi"] \arrow[dr] & & G \arrow[dl] \\
& M &
\end{tikzcd}$$

commutes. *isomorphisms* between fiber bundles and *automorphisms* of a fiber bundle are defined in the obvious way.

Remark 2.8. If $x = (x^1, \dots, x^n)$ are coordinates on M, $u = (u^1, \dots, u^m)$ are fiber coordinates on E and $v = (v_1, \dots, v_p)$ are fiber coordinates on G, then, locally, F is completely determined by functions

$$v_a = F_a(x, u) = F^*(v^a), \quad a = 1, \dots, p.$$

We write

$$\Phi : v_a = F_a(x, u).$$

If s is a section of E, locally given by

$$s : u^\alpha = s^\alpha(x),$$

then $F \circ s$ is a section of G locally given by

$$\Phi \circ s : v_a = F_a(s(x)).$$

2.3. **Jets of sections of a fiber bundle.** In order to define PDEs in a coordinate independent way, we need a coordinate independent definition of *multiple partial derivatives*. This is provided by jets (of sections of a fiber bundle).

Let $E \to M$ be a fiber bundle. We will interpret the base coordinates $x = (x^1, \dots, x^n)$ as independent variables and the fiber coordinates $u = (u^1, \dots, u^m)$ as dependent variables. Given a local section s of E defined around a point $x_0 \in M$ what are the k-th order (*multiple, partial*) derivatives of s? We first define when do two sections s, t, defined around the same point x_0, have the same partial derivatives up to order k at x_0.

Proposition 2.9. *Let s, t be local sections of E defined around $x_0 \in M$. Assume $s(x_0) = t(x_0) =: e_0$ and let (x, u) be bundle coordinates around e_0 such that, locally*

$$s : u^\alpha = s^\alpha(x), \quad \textit{and} \quad t : u^\alpha = t^\alpha(x).$$

Denote by $\mu_{x_0} \subseteq C^\infty(M)$ the ideal consisting of functions h such that $h(x_0) = 0$. For every $k \geq 0$, the following two conditions are equivalent:

(1) *for all I such that $|I| \leq k$*
$$s^\alpha{}_{,I}(x_0) = t^\alpha{}_{,I}(x_0);$$
(2) *if $f \in C^\infty(E)$ then*
$$f \circ s \in \mu_{x_0}^{k+1} \quad \textit{iff} \quad f \circ t \in \mu_{x_0}^{k+1}.$$

Exercise 2.10. Prove Proposition 2.9 (*Hint: recall that a function $h \in C^\infty(M)$ is in $\mu_{x_0}^k$ iff its derivatives up to order k vanish in one, hence any, coordinate system (which is an immediate consequence of the Hadamard's Lemma)*).

Definition 2.11. Two local sections s, t defined around $x_0 \in M$ are *tangent up to order $k \geq 0$ at x_0* if $s(x_0) = t(x_0)$, and one of the equivalent conditions of Proposition 2.9 is satisfied. In this case we write $s \sim_{x_0}^k t$.

Remark 2.12.
s, t are tangent up to order 0 if $s(x_0) = t(x_0)$.
s, t are tangent up to order 1 if they are tangent up to order 0 and, additionally, $d_{x_0}s = d_{x_0}t$, or, which is the same, their graphs are tangent at $s(x_0) = t(x_0)$.

Tangency up to order k at x_0 is an equivalence relation on the set of all local sections defined around x_0. The equivalence class of s is denoted $j^k_{x_0}s$ and called the *k-jet* of s at x_0. In practice it contains a full information on "derivatives of s" up to order k at x_0.

Denote by $J^k_{x_0}E$ the space of all k-jets of sections of E at x_0 and

$$J^k E := \bigsqcup_{x_0 \in M} J^k_{x_0}E.$$

J^kE is the *k-jet space*, or *k-jet bundle*, of E. In practice, points in J^kE are Taylor polynomials of sections of E at all possible points of M.

There are obvious surjections $p : J^kE \to J^{k-1}E$, $j^k_x s \mapsto j^{k-1}_x s$ which consist in "forgetting the last derivative".

There are also surjections, the *source maps*, $\pi : J^kE \to M$, $j^k_x s \mapsto x$.

J^0E identifies with E, under $j^0_x s \mapsto s(x)$. So $p \circ \cdots \circ p : J^kE \to J^0E$ identifies with the *target map* $p_E : J^kE \to E$, $j^k_x s \mapsto s(x)$.

Remark 2.13. We can put coordinates on J^kE as follows. Let (x, u) be bundle coordinates on E, defined in a neighborhood U and let $U_k := p_E^{-1}(U) \subseteq J^kE$. Let $z = j^k_x s \in U_k$, with $s : u^\alpha = s^\alpha(x)$, and put

$$u^\alpha_I(z) = s^\alpha{}_{,I}(x), \quad \alpha = 1, \ldots, m, \quad I \text{ a multi-index s.t. } |I| \leq k.$$

Proposition 2.14. *$(U_k, (x, \ldots, u_I, \ldots))$ is a chart on J^kE (called a* standard chart*). Any two such charts are compatible. With this atlas, J^kE is a smooth manifold,*

$$\dim J^kE = n + m\binom{n+k}{k},$$

and both $p : J^kE \to J^{k-1}E$ and $p_E : J^kE \to E$ are fiber bundles with abstract fiber diffeomorphic to some Euclidean space.

Exercise 2.15. Prove Proposition 2.14 (*Hint: prove, by induction on k, that the transition maps between standard charts are polynomials in derivatives u_I, $0 < |I| \leq k$. For the fiber of p notice that, for every $e \in E$, and every sequence $S = (s_I)_{|I|>0} \subseteq \mathbb{R}^m$, S is the sequence of derivatives of a section of E through e*).

Exercise 2.16. Assume $E \to M$ is a vector bundle. Show that $J^kE \to M$ is a vector bundle as well (*Hint: notice that, for all $x \in M$ the fiber J^k_xE of J^kE over x is a vector space with the following operations:*

$$j^k_x s + j^k_x t = j^k_x(s + t)$$
$$r \cdot j^k_x s = j^k_x(rs)$$

s, t *local sections of* E *and* $r \in \mathbb{R}$.)

We can also use the jet space to encode *derivative functions*. Let s be a section of E.

Definition 2.17. The *k-jet prolongation of* s is the following smooth section of $J^k E \to M$:

$$j^k s : M \to J^k E, \quad x \mapsto j^k_x s.$$

Exercise 2.18. Show that $j^k s$ is actually a smooth section (*Hint: show that if* $s : u^\alpha = s^\alpha(x)$ *then*

$$j^k s : u^\alpha_I = s^\alpha{}_{,I}(x),$$

hence it is locally smooth).

3. The Cartan distribution

3.1. **Distributions on manifolds.** The jet space $J^k E$ is equipped with a canonical structure called the *Cartan distribution*, which is a(n other) manifestation of the fact that $J^k E$ is a space of derivatives.

Let N be a smooth n-dimensional manifold

Definition 3.1. A rank k (regular) distribution on N is a rank k vector subbundle D of the tangent bundle TN.

Remark 3.2. In other words a rank k distribution is the datum of a k-dimensional subspace $D_z \subseteq T_z N$ for any $z \in N$ in such a way that D_z depends smoothly on z. Equivalently, locally there are k vector fields $Y_1, \ldots, Y_k$ such that

$$D_z = \langle Y_1|_z, \ldots, Y_k|_z \rangle$$

for all z. In this case we also write

$$D = \langle Y_1, \ldots, Y_k \rangle,$$

and say that D *is (locally) spanned by* $Y_1, \ldots, Y_k$. In a dual way, locally there are $n-k$ differential 1-forms $\omega_1, \ldots, \omega_{n-k}$ such that

$$D_z = \ker \omega_1|_z \cap \cdots \cap \ker \omega_{n-k}|_z.$$

There is a dual approach to distributions. Given a rank k distribution $D \subseteq TN$, it is natural to consider the *normal bundle*, i.e. the quotient bundle $V = TM/D$, whose fiber over z is $T_z N / D_z$. So there is a canonical projection

$$\theta : TM \to V, \quad v \mapsto v \bmod D,$$

with kernel D. θ can be seen as a differential 1-form with values in V:

$$\theta \in \Omega^1(N, V) = \Omega^1(N) \otimes \Gamma(V),$$

where the tensor product is over $C^\infty(N)$, and it is called the *structure* 1*-form of* D.

Conversely, given a surjective 1-form θ on N with values in a rank $n-k$ vector bundle V,

(1) the kernel of θ is a rank k distribution D on N,
(2) V identifies canonically with TN/D, and
(3) θ identifies with the projection $TN \to TN/D$.

This shows that distributions are basically equivalent to surjective vector bundle valued 1-forms.

Definition 3.3. A connected (immersed) submanifold $S \subseteq N$ is an *integral submanifold* of a distribution D on N, if $T_zS \subseteq D_z$ for all $z \in S$. An integral submanifold S is *locally maximal* if, for every $z \in S$, and every neighborhood U of z, $S \cap U$ is not contained in any integral submanifold of bigger dimension.

Remark 3.4. A distribution D possesses at least integral *curves* but needs not to possess integral manifolds of the same dimension as $\operatorname{rank} D$. On the other hand it may possess several locally maximal integral submanifolds through the same point, even of different dimensions. See below for a distinguished example.

Example 3.5. In $\mathbb{R}^n$ with cartesian coordinates $(x^1, \ldots, x^n)$ consider the distribution D spanned by the first k coordinate vector fields:

$$D = \left\langle \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k} \right\rangle.$$

The affine subspaces

$$\begin{cases} x^{k+1} = c^1 \\ \quad \cdots \\ x^n = c^{n-k} \end{cases} \qquad (c^1, \ldots, c^{n-k}) = \mathrm{const}$$

are all locally maximal (actually even globally maximal) integral submanifolds of dimension $k = \operatorname{rank} D$. Every locally maximal integral submanifold is a connected open subset in one of those and vice-versa.

Let D be a distribution and let $\theta \in \Omega^1(N, V)$ be its structure 1-form. There is a skew-symmetric, $\mathbb{R}$-bilinear map

$$\omega : \Gamma(D) \times \Gamma(D) \to \Gamma(V), \quad (X, Y) \mapsto \theta([X, Y]) = [X, Y] \bmod \Gamma(D)$$

Exercise 3.6. Show that ω is $C^\infty(N)$-bilinear and conclude that it comes from a vector bundle map, also denoted ω:

$$\omega : \wedge^2 D \to V.$$

Definition 3.7. ω is called the *curvature* of D.
If $\omega = 0$ we say that D is *involutive* or *integrable*.
If ω is full-rank we say that D is *maximally non-integrable*.

Notice that D is integrable iff its section $\Gamma(D)$ are preserved by the commutator, i.e. $[X, Y] \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$. The terminology is motivated by the following

Theorem 3.8 (Frobenius)**.** *D is integrable iff locally, around every point of N, there are coordinates $(x^1, \ldots, x^n)$ such that*

$$D = \left\langle \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k} \right\rangle.$$

Frobenius Theorem immediately implies that an integrable distribution D possesses locally maximal integral submanifolds of the same dimension as $\operatorname{rank} D$ through any point.

Let D be a distribution on a manifold N.

Definition 3.9. A *symmetry* of D is a diffeomorphism $\phi : N \to N$ preserving D, i.e. $d\phi(D) = D$. An *infinitesimal symmetry* of D is a vector field X on N generating a flow by symmetries. A *characteristic symmetry* of D is an infinitesimal symmetry X which is, additionally, in D, i.e. $X \in \Gamma(D)$.

Denote by $D\Omega^1 \subseteq \Omega^1(N)$ the submodule consisting of differential 1-forms annihilating vector fields in $\Gamma(D)$:

$$\omega \in D\Omega^1 \;\Leftrightarrow\; \omega(X) = 0 \text{ for all } X \in \Gamma(D).$$

Exercise 3.10. Show that

(1) symmetries form a group under composition;
(2) symmetries of a distribution D map (locally maximal) integral submanifolds to (locally maximal) integral submanifolds;
(3) a diffeomorphism $\phi : N \to N$ is a symmetry iff $\phi^*(X) \subseteq \Gamma(D)$ for all $X \in \Gamma(D)$
(4) a diffeomorphism $\phi : N \to N$ is a symmetry iff $\phi^*(\omega) \in D\Omega^1$ for all $\omega \in D\Omega^1$.

Exercise 3.11. Show that

(1) infinitesimal symmetries form a Lie algebra (denoted $\mathfrak{X}_D$) under the commutator;
(2) characteristic symmetries form an ideal in $\mathfrak{X}_D$;
(3) the flowout of an integral manifold under a characteristic symmetry is an integral manifold;
(4) a vector field X is an infinitesimal symmetry iff $[X, Y] \in \Gamma(D)$ for all $Y \in \Gamma(D)$;
(5) a vector field X is an infinitesimal symmetry iff $\mathcal{L}_X \omega \in D\Omega^1$ for all $\omega \in D\Omega^1$.

(*Hint: for (4) use formula*

$$\phi_t^*[X, Y] = \frac{d}{dt} \phi_t^*(Y),$$

for all $X, Y \in \mathfrak{X}(N)$, where $\{\phi_t\}$ is the flow of X. For (5) use (4) and Cartan calculus).

Exercise 3.12. Show that

(1) a distribution D is integrable iff every section of D is a characteristic symmetry;

(2) If D is maximally non-integrable then it does not possess characteristic symmetries. If, additionally, $\operatorname{rank}\omega = \mathrm{const}$, then the converse if also true. (*Hint for the second part of (2): if* $\operatorname{rank}\omega = \mathrm{const}$*, then tangent vectors* v *such that* $\omega(v, -) = 0$ *form a(n integrable) distribution* $K \subseteq D|_U$*. Show that characteristic symmetries of* D *are precisely sections of* K*.*)

3.2. **The Cartan distribution and its structure 1-form.** Now let's go back to the k-jet space $J^k E$ of some fiber bundle. We want to show that $J^k E$ is equipped with a canonical, maximally non-integrable distribution: the *Cartan distribution* C. There are several equivalent ways to define C. First we define *jet-planes*.

Let $z \in J^k E$, and let $x = \pi(z)$. Then $z = j^k_x s$ for some, non unique, local section of E defined around x.

Definition 3.13. A *jet-plane* at z is any n-dimensional subspace $J \subseteq T_z(J^k E)$ of the form

$$J = T_z(\text{graph of } j^k s), \quad \text{for some representative } s \text{ of } z.$$

In other words a jet-plane is the tangent space to the graph of a jet prolongation.

Remark 3.14. Notice that the condition that s is a representative of z is equivalent to the condition that z is a point in the graph of $j^k s$, and this makes jet-planes well-defined.

Proposition 3.15. *jet-planes at* $z \in J^k E$ *are parameterized by points in* $p^{-1}(z) \subseteq J^{k+1}E$*, i.e. there is a (actually canonical) bijection*

$$b : p^{-1}(z) \to \{\text{jet-planes at } z\}.$$

Proof. First of all there is an informal argument which explains the statement. Namely, z is, morally, a Taylor polynomial of order k, and a representative s of it is a section whose Taylor polynomial of order k is exactly z. Now $j^k s$ encodes derivatives of s up to order k, so the tangent space at z to the graph of $j^k s$ encodes derivatives of s at $\pi(z)$ up to order $k+1$. This is precisely the same information contained in a point in $p^{-1}(z) \subseteq J^{k+1}$.

More rigorously, we define b as follows. Put $\overline{x} = \pi(z)$. Since $p(j^{k+1}_{\overline{x}} s) = j^k_{\overline{x}} s$, a point $j^{k+1}_{\overline{x}} s \in J^{k+1}E$ is in $p^{-1}(z)$ iff s is a representative of z, i.e. $z = j^k_{\overline{x}} s$. This means that z is in the graph of $j^k s$. So it makes sense putting

$$b(j^{k+1}_{\overline{x}} s) := T_z(\text{graph of } j^k s).$$

$\square$

Exercise 3.16. Conclude the proof of Proposition 3.15 showing that b is a well-defined bijection. (*Hint: use local coordinates.*)

It follows from the arbitrariness of k that points in $J^k E$ correspond bijectively to jet-planes in $J^{k-1}E$. Let $z \in J^k E$, and $\underline{z} = p(z)$. In the following, we denote by

$$J_z =: b(z) \subseteq T_{\underline{z}}(J^{k-1}E)$$

the corresponding jet-plane.

Definition 3.17. The *Cartan plane* at z is

$$C_z := (dp)^{-1}(J_z) \subseteq T_z(J^k E).$$

The *Cartan distribution* is

$$C : z \mapsto C_z.$$

Proposition 3.18. *The Cartan distribution is a smooth distribution, and*

$$\operatorname{rank} C = n + m\binom{n+k-1}{k}.$$

Locally

$$C = \left\langle \dots, D_i, \dots, \frac{\partial}{\partial u^\alpha_K}, \dots \right\rangle_{|K|=k},$$

where

$$D_i = \frac{\partial}{\partial x^i} + \sum_{|I|<k} u^\alpha_{Ii} \frac{\partial}{\partial u^\alpha_I}, \quad i = 1, \dots, n.$$

Definition 3.19. Vector fields D_i are called the (truncated) *total derivatives.*

Exercise 3.20. Prove Proposition 3.18. (*Hint: Prove that, for all $z \in J^k E$, C_z is spanned by*

$$D_i|_z \quad \text{and} \quad \left.\frac{\partial}{\partial u^\alpha_K}\right|_z.$$

To do this notice that the $\left.\frac{\partial}{\partial u^\alpha_K}\right|_z$'s belong to C_z, and that the $(dp)(D_i|_z)$'s form a basis in J_z. Then use linear algebra.)

Exercise 3.21. Prove that the Cartan plane at z is spanned by all jet-planes at z (this provides an alternative definition of the Cartan distribution) (*Hint: use local coordinates*).

We now provide a dual description of the Cartan distribution.

Definition 3.22. The *vertical bundle* to a fiber bundle $\pi : E \to M$ is the distribution $VE = \ker d\pi$ consisting of tangent spaces to fibers of π

Consider the k-jet bundle $J^k E$ and let $V := V(J^{k-1}E) \to J^{k-1}E$ the vertical bundle to the $(k-1)$-jet bundle $J^{k-1}E \to M$. We will also consider the pull-back $p^*V \to J^k E$ which is the vector bundle whose fiber over $z \in J^k E$ is the fiber of V over $p(z)$.

Proposition 3.23. *The Cartan distribution C is the kernel of a canonical p^*V-valued 1-form θ. Locally*

$$\theta = \sum_{|I|<k} \left(du^\alpha_I - u^\alpha_{Ii} dx^i\right) \otimes \frac{\partial}{\partial u^\alpha_I}. \tag{1}$$

Proof. Define θ as follows. For $z = j^k s \in J^k E$, and $v \in T_z(J^k E)$, put

$$\theta(v) = dp(v) - d(j^{k-1}s \circ \pi)(v) \in T_{p(z)} J^{k-1}E,$$

which is independent of the choice of s. Additionally,

$$d\pi(\theta(v)) = d\pi \circ dp(v) - d\pi \circ d(j^{k-1}s \circ \pi)(v) = d\pi(v) - d\pi(v) = 0,$$

so that $\theta(v) \in V_{p(z)}$. Locally, θ is given by (1) and this shows that θ is a smooth p^*V-valued 1-form (Exercise 3.24). Finally, let $v \in \ker \theta$ then

$$dp(v) = d(j^{k-1}s \circ \pi)(v) = dj^{k-1}s \circ d\pi(v) \in J_z$$

so that $v \in C_z$. Conversely, let $v \in C_z$, then $dp(v) \in J_z$. As J_z projects isomorphically to $T_{\pi(z)}M$ under $d\pi$ we have

$$dp(v) = dj^{k-1}s \circ d\pi(v) = d(j^{k-1}s \circ \pi)(v),$$

hence $\theta(v) = 0$. □

Exercise 3.24. Complete the proof of Proposition 3.23 showing that θ is well-defined and locally given by (1).

Remark 3.25. It follows from Proposition 3.23 that C is locally the intersection of the kernels of the 1-forms

$$\omega_I^\alpha := du_I^\alpha - u_{Ii}^\alpha dx^i, \quad |I| < k.$$

The latter are sometimes called *Cartan forms.* A 1-form vanishes on C iff it is locally a linear combination of Cartan forms.

Remark 3.26. Let $\pi : E \to M$ be a fiber bundle, and $e \in E$. A subspace $W \subseteq T_e E$ is *π-horizontal* if $d\pi$ maps it injectively to $T_{\pi(e)}M$. For instance jet-planes are π-horizontal subspaces. Even more, let $z \in J^k E$, then

$$T_{p(z)}(J^{k-1}E) = J_z \oplus V_{p(z)}$$

and there is an alternative description of θ. Namely, denote by

$$\mathrm{pr}_z : T_{p(z)}(J^{k-1}E) \to V_{p(z)}$$

the projection with kernel J_z. Then, for $v \in T_z(J^k E)$,

$$\theta(v) = \mathrm{pr}_z \circ dp(v).$$

In particular θ is surjective, the quotient bundle $T(J^k E)/C$ identifies canonically with p^*V, and, in this identification, θ is the structure 1-form of C.

Surjectivity of θ is also clear from (1).

Proposition 3.27. *The curvature*

$$\omega : \wedge^2 C \to p^*V$$

of the Cartan distribution is locally given by

$$\omega = \sum_{|I|=k-1} \left(du^\alpha_{Ii} \wedge dx^i\right)\big|_C \otimes \frac{\partial}{\partial u^\alpha_I}$$

Corollary 3.28. *The Cartan distribution is maximally non-integrable.*

Exercise 3.29. Prove Proposition 3.27 and Corollary 3.28.

Let $z \in J^k E$, and $x = \pi(z)$. Recall that n-dimensional, π-horizontal subspaces of $T_z(J^k E)$ form an affine space modelled over $T^*_x M \otimes V_z(J^k E)$.

Problem 3.30. *Prove that jet-planes can be characterized as n-dimensional, π-horizontal subspaces J that are isotropic wrt ω, i.e. $\omega|_J = 0$. Use this to show that jet-planes form an affine subspace in the affine space of all n-dimensional, π-horizontal subspaces of $T_z(J^k E)$. Finally, prove that $p : J^{k+1}E \to J^k E$ is an affine bundle modelled over*

$$\pi^*(S^{k+1}T^*M) \otimes p_E^*(VE).$$

The Cartan distribution detects jet prolongations in the sense of the following

Proposition 3.31. *The graph of a section σ of $J^k E \to M$ is an integral submanifold of the Cartan distribution iff $\sigma = j^k s_\sigma$ with $s_\sigma = p_E \circ \sigma$.*

Proof. Use local coordinates. Locally $\sigma : u^\alpha_I = \sigma^\alpha_I$. Now, the graph of σ is an integral submanifold iff

$$0 = \sigma^*(\omega^\alpha_I) = d\sigma^\alpha_I - \sigma^\alpha_{Ii} dx^i,$$

i.e.,

$$\sigma^\alpha_I{}_{,i} = \sigma^\alpha_{Ii},$$

and, by induction on $|I|$,

$$\sigma^\alpha_I = \sigma^\alpha{}_{,I}\,,$$

which concludes the proof. □

Recall that a submanifold $S \subseteq J^k E$ is π-horizontal iff $T_z S$ is a π-horizontal subspace for all $z \in S$.

Corollary 3.32. *An n-dimensional, π-horizontal submanifold $S \subseteq J^k E$ is an integral submanifold of the Cartan distribution iff it is locally the graph of a jet prolongation.*

Proof. Every π-horizontal submanifold of $J^k E$ is locally the graph of a section. □

3.3. Locally maximal integral submanifolds of the Cartan distribution.

Remark 3.33. Let $S \subseteq J^k E$ be an integral submanifold of C. For simplicity, assume that $p(S)$ is a submanifold (and $p : S \to p(S)$ a submersion). Then every tangent space to $p(S)$ is contained into some jet-plane. Hence $p(S)$ is a π-horizontal integral manifold. We will now show that, if S is locally maximal (among integral manifolds) then it is completely determined by $p(S)$ via the *ray construction*.

First of all notice that

1. graph of jet prolongations, and
2. fibers of $p : J^k E \to J^{k-1} E$,

are integral submanifolds of the Cartan distribution.

Exercise 3.34. Let $S \subseteq J^k E$ be a π-horizontal (non-necessarily locally maximal) integral submanifold of the Cartan distribution. Prove that, locally around every point, S is contained into the graph of a jet prolongation. (*Hint: notice that $\pi : S \to M$ is an immersion. Choose a neighborhood $U \subseteq S$, such that $\pi(U) \subseteq M$ is an embedded submanifold and $\pi : U \to \pi(U)$ is a diffeomorphism. Denote $V := p_E(U) \subseteq E$ and notice that V is an embedded submanifold such that $\pi : V \to \pi(U)$ is a diffeomeorphism. Extend $\pi^{-1} : \pi(U) \to V$ to a local section s of $E \to M$, and prove that the graph of $j^k s$ contains U. Use local coordinates.*)

Let $z \in J^k E$, and $\underline{z} := p(z)$. Recall that we denote by $J_z \subseteq T_{\underline{z}}(J^{k-1}E)$ the jet-plane corresponding to z. Now, let $H \subseteq C_{\underline{z}}$ be a π-horizontal subspace of dimension h (then $h \leq n$).

Definition 3.35. The *ray* $\ell(H)$ of H is the subspace in $p^{-1}(\underline{z})$ given by

$$\ell(H) := \{z \in p^{-1}(\underline{z}) : J_z \supseteq H\}.$$

Remark 3.36. If $H_1 \subseteq H_2 \subseteq C_{\underline{z}}$ are two π-horizontal subspaces then $\ell(H_2) \subseteq \ell(H_1)$.

Example 3.37.

1. $H = 0 \Rightarrow \ell(H) = p^{-1}(\underline{z})$.
2. $H = J_z \Rightarrow \ell(H) = \{z\}$.

Exercise 3.38. Prove that

1. $\ell(H)$ is non-empty iff H is ω-isotropic, i.e. $\omega|_H = 0$;
2. if H is ω-isotropic, then $\ell(H) \subseteq p^{-1}(\underline{z})$ is an affine subspace of dimension

$$\dim \ell(H) = m \binom{n-h+k-1}{k}. \tag{2}$$

Now, let $S \subseteq J^{k-1}E$ be an h-dimensional, π-horizontal, integral manifold of the Cartan distribution.

Exercise 3.39. Prove that $T_{\underline{z}} S$ is isotropic wrt ω for all $\underline{z} \in S$.

Definition 3.40. The *ray* $\ell(S)$ of S is the subset in J^kE given by

$$\ell(S) := \{z \in J^kE : J_z \supseteq T_{p(z)}S\} = \bigsqcup_{\underline{z} \in S} \ell(T_{\underline{z}}S).$$

Example 3.41.

(1) $S = \{\underline{z}\} \Rightarrow \ell(S) = p^{-1}(\underline{z})$.
(2) $S = \text{graph of } j^{k-1}s \Rightarrow \ell(S) = \text{graph of } j^k s$.

In both cases $\ell(S)$ is an integral submanifold of the Cartan distribution.

Theorem 3.42. *$\ell(S) \subseteq J^kE$ is an affine bundle over S, and a locally maximal integral submanifold of the Cartan distribution of dimension*

$$\dim \ell(S) = h + m\binom{n-h+k-1}{k}. \tag{3}$$

Every locally maximal integral submanifold of the Cartan distribution on J^kE is almost everywhere, locally, of this kind.

Proof. We leave as an Exercise 3.43 to prove that $\ell(S)$ is an affine bundle over S. Formula (3) then follows from (2). To see that $\ell(S)$ is an integral manifold, notice that for every point $z \in \ell(S)$

$$dp(T_z\ell(S)) = T_{p(z)}S \subseteq J_z.$$

so $T_z\ell(S) \subseteq C_z$. Next we prove that every locally maximal integral submanifold N is almost everywhere, locally, of the form $\ell(S)$. So, let $N \subseteq J^kE$ be a locally maximal integral submanifold of C. The rank of $p : N \to J^{k-1}E$ is almost everywhere, locally constant. So, without loss of generality, we may assume that $S := p(N) \subseteq J^{k-1}E$ is an integral manifold and $p : N \to S$ is a submersion. Notice that, as N is an integral manifold, S is an integral manifold as well, and, additionally, S is π-horizontal. We want to show that $N = \ell(S)$. So let $z \in N$, then

$$J_z = dp(C_z) \supseteq dp(T_zN) = T_{p(z)}S.$$

This shows that $N \subseteq \ell(S)$. As N is locally maximal then $N = \ell(S)$.

It remains to prove that for every S, $\ell(S)$ is locally maximal. So, suppose that $\ell(S) \subseteq N$ for some integral manifold N. Then $p(N) \supseteq S$. Assume, preliminarily, that the rank of $p : N \to J^{k-1}E$ is constant around the points of $\ell(S)$. Then $N = \ell(S')$ around points of S, and $S' \supseteq S$. It now follows from (3) that $\dim N = \dim \ell(S') \leq \dim \ell(S)$, so $\ell(S)$ coincides (locally) with N and it is locally maximal itself. Finally notice that if the rank of $p : N \to J^{k-1}E$ was not constant along $\ell(S)$ then, for similar reasons, "it would increase along $\ell(S)$" which is impossible. We leave details to the reader. $\square$

Exercise 3.43. Prove that $\ell(S)$ is an affine bundle over S.

Exercise 3.44. Prove that the rank of $p : N \to J^{k-1}E$ (in the last part of the proof of Theorem 3.42) is necessarily constant around $\ell(S)$ (*Hint: remember that the rank of a smooth map is lower semicontinuous*).

Remark 3.45. It follows from Theorem 3.42 that, through any point in a jet space, there can be several locally maximal integral submanifold (not even agreeing locally and) not even of the same dimension.

Corollary 3.46.

(1) *If $m = n = 1$ (ODEs in 1 dependent variable) then all locally maximal integral submanifolds of the Cartan distribution share the same dimension 1.*
(2) *If $m = k = 1$ (first order PDEs in 1 dependent variable) then all locally maximal integral submanifolds of the Cartan distribution share the same dimension n.*
(3) *In all other cases (open subsets in) the fibers of p are locally maximal integral submanifolds of the maximum possible dimension.*

4. Symmetries of the Cartan distribution

Recall that a symmetry of the Cartan distribution on J^kE is a diffeomorphism $\phi : J^kE \to J^kE$ such that $d\phi(C) = C$.

Definition 4.1. A *Lie transformation* of J^kE is a symmetry of the Cartan distribution on J^kE. An *infinitesimal Lie transformation* is an infinitesimal simmetry of C.

If $k = 0$, then $J^kE = E$ and $C = TE$. So every diffeomorphism $\phi : E \to E$ is a Lie transformation. We call it a *point transformation*.

Exercise 4.2. Let D be a distribution with curvature ω. Prove that a symmetry of D maps ω-isotropic subspaces to ω-isotropic subspace.

Given a Lie transformation ϕ of J^kE, we can construct a Lie transformation of $J^{k+1}E$ as follows. Let J be a jet-plane at $z \in J^kE$. So $J = J_{z'}$ for some $z' \in p^{-1}(z)$. In view of Exercise 4.2, $d\phi(J)$ is an n-dimensional, ω-isotropic subspace at $\phi(z)$. If it is also π-horizontal, then it is a jet-plane. As being π-horizontal is an open condition, this happens for almost all $z' \in J^{k+1}E$, i.e. on an open dense subset $U \subseteq J^{k+1}E$. Define

$$\phi^{(1)} : U \to J^{k+1}E,$$

letting $\phi^{(1)}(z')$ be implicitly given by

$$J_{\phi^{(1)}(z')} = d\phi(J_{z'}).$$

Proposition 4.3. $\phi^{(1)}$ *is a (local) Lie transformation (of $J^{k+1}E$).*

Proof. We leave it to the reader checking that $\phi^{(1)}$ is a smooth map. It is clearly a local diffeomorphism: $(\phi^{-1})^{(1)}$ provides the inverse. Finally, prove that $\phi^{(1)}$ preserves the Cartan distribution. First of all notice that the diagram

$$\begin{tikzcd} U \arrow[r, "\phi^{(1)}"] \arrow[d] & J^{k+1}E \arrow[d] \\ J^kE \arrow[r, "\phi"] & J^kE \end{tikzcd}$$

commutes. Hence $dp \circ d\phi^{(1)} = d\phi \circ dp$. So, for all $z' \in U$

$$dp \circ d\phi^{(1)}(C_{z'}) = d\phi \circ dp(C_{z'}) = d\phi(J_{z'}) = J_{\phi^{(1)}(z')}.$$

This shows that $d\phi^{(1)}(C_{z'}) \subseteq C_{\phi^{(1)}(z')}$ as claimed. □

Definition 4.4. The Lie transformation $\phi^{(1)}$ is called the (first) *prolongation* of ϕ.

In the following, given a Lie transformation ϕ of $J^k E$ we put

$$\phi^{(r)} := \phi^{\underbrace{(1)(1)\cdots(1)}_{r \text{ times}}}$$

Theorem 4.5 (Lie-Bäcklund)**.**

(1) *Let ψ be a (local) Lie transformation of $J^k E$, then locally $\psi = \phi_1^{(k-1)}$ for some, necessarily unique, Lie transformation ϕ_1 of $J^1 E$. The correspondence $\psi \mapsto \phi_1$ is one-to-one and group property preserving.*

(2) *In the additional hypothesis that $m \neq 1$, then, even more, $\psi = \phi^{(k)}$ for some point transformation ϕ. The correspondence $\psi \mapsto \phi$ is one-to-one and group property preserving.*

Proof. Assume $m, n > 1$. The case $m = n = 1$ (ODEs in 1 dependent variable) requires an *ad hoc proof* which we omit (see Lemma 3.2 in[1]). Let ψ be a (local) Lie transformation of $J^k E$, with $k > 1$. From Exercise 3.10, ψ maps locally maximal integral submanifolds to locally maximal integral submanifolds. As it also preserves dimensions, it maps locally maximal integral submanifolds of the maximum possible dimension to locally maximal integral submanifolds. We assumed $m, n, k > 1$ so, according to Corollary 3.46, locally maximal integral submanifolds are (open submanifolds in) fibers of p. So, locally, ψ maps fibers to fibers and it descends to a (local) diffeomorphism $\psi_{(1)}$ of $J^{k-1}E$. We want to show that $\psi_{(1)}$ is a Lie transformation. By construction, diagram

$$\begin{tikzcd} J^k E \arrow[r, "\psi"] \arrow[d] & J^k E \arrow[d] \\ J^{k-1} E \arrow[r, "\psi_{(1)}"] & J^{k-1} E \end{tikzcd}$$

commutes, so, for all $z \in J^k E$,

$$d\psi_{(1)}(J_z) = d\psi_{(1)} \circ dp(C_z) = dp \circ d\psi(C_z) = dp(C_{\psi(z)}) = J_{\psi(z)}. \tag{4}$$

This shows that $\psi_{(1)}$ maps jet-planes to jet-planes. Since jet-planes span Cartan planes, it follows that $\psi_{(1)}$ preserves the Cartan distribution, i.e. it is a Lie transformation. Computation (4) does also show that $\psi = {\psi_{(1)}}^{(1)}$. The first item in the statement can now be proved by induction.

[1]A. V. Bocharov et al., Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, *Transl. Math. Mon.* **182**, Amer. Math. Soc., Providence, 1999.

For the second item, notice that, if $m > 1$, then the same argument as above shows that a Lie transformation of J^1E is the prolongation of a point transformation. $\square$

Example 4.6. We want to find coordinate formulas for the first prolongation of a point transformation. So let $\phi : E \to E$ be a point transformation locally given by

$$\begin{array}{r}\phi^*(x^i) = \mathcal{X}^i = \mathcal{X}^i(x,u) \\ \phi^*(u^\alpha) = \mathcal{U}^\alpha = \mathcal{U}^\alpha(x,u)\end{array}. \tag{5}$$

As $\phi^{(1)}$ projects onto ϕ, it is locally given by

$$\begin{array}{r}\phi^*(x^i) = \mathcal{X}^i \\ \phi^*(u^\alpha) = \mathcal{U}^\alpha \\ \phi^*(u^\alpha_i) = \mathcal{U}^\alpha_i = \mathcal{U}^\alpha_i(x,u,u')\end{array}, \tag{6}$$

where the u' are partial derivatives of the u, and we want to find the $\mathcal{U}^\alpha_i$ in terms of the $\mathcal{X}^i$ and the $\mathcal{U}^\alpha$. In view of the Lie-Bäcklund Theorem, the $\mathcal{U}^\alpha_i$ are completely determined by the condition that $\phi^{(1)}$ is a Lie transformation. From Exercise 3.10.(3) this is equivalent to $\phi^{(1)*}(\omega^\alpha) \in C\Omega^1$, $\alpha = 1,\dots,m$, i.e.

$$\phi^{(1)*}(\omega^\alpha)\,(D_j) = \phi^{(1)*}(\omega^\alpha)\left(\frac{\partial}{\partial u^\beta_j}\right) = 0.$$

Now

$$\phi^{(1)*}(\omega^\alpha) = \phi^{(1)*}(du^\alpha - u^\alpha_i dx^i) = d\mathcal{U}^\alpha - \mathcal{U}^\alpha_i d\mathcal{X}^i.$$

As the $\mathcal{X}^i$ and the $\mathcal{U}^\alpha$ are functions of the only (x,u), $\phi^{(1)*}(\omega^\alpha)$ annihilates the $\partial/\partial u^\beta_j$ identically, and it remains to check when does $\phi^{(1)*}(\omega^\alpha)\,(D_j)$ vanish. We have

$$\phi^{(1)*}(\omega^\alpha)\,(D_j) = D_j\mathcal{U}^\alpha - \mathcal{U}^\alpha_i D_j\mathcal{X}^i.$$

We conclude that (6) defines a Lie transformation iff

$$\mathcal{U}^\alpha_i D_j\mathcal{X}^i = D_j\mathcal{U}^\alpha.$$

As ϕ is a diffeomorphism, the matrix $(D_j\mathcal{X}^i)$ is invertible in an open and dense subset of J^1E (Exercise 4.7), and $\phi^{(1)}$ is only defined on the open and dense subset U where $(D_j\mathcal{X}^i)$ is invertible. If $(\mathcal{Y}^j_i)$ is the inverse matrix, $\mathcal{Y}^j_i = \mathcal{Y}^j_i(x,u,u')$, we have

$$\mathcal{U}^\alpha_i = \mathcal{Y}^j_i \cdot D_j\mathcal{U}^\alpha.$$

Exercise 4.7. Prove that the matrix $(D_j\mathcal{X}^i)$ in Example 4.6 is invertible in an open and dense subset.

Exercise 4.8. Let $\phi : E \to E$ be a point transformation locally given by (5). Find coordinate formulas for the second prolongation $\phi^{(2)}$.

Remark 4.9. The Lie-Bäcklund Theorem cannot be improved extending the second item to the case $m = 1$ as the following counter-example shows: let $M = \mathbb{R}^n$, and

$E = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Then $J^1E \simeq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and (x, u, u') are global coordinates on it. It is easy to check (Exercise 4.10) that the *Legendre transformation*:

$$\psi : J^1E \to J^1E, \quad (x, u, u') \mapsto (u', x \cdot u' - u, x) \tag{7}$$

is a Lie transformation. As it does not preserve the fibers of $p : J^1E \to E$, it cannot be the prolongation of any point transformation. We conclude that the case $k = m = 1$ is somewhat peculiar and, to some extent, should be studied separately. This is the aim of *contact geometry* and will be pursued in Section 7.

Exercise 4.10. Prove that the Legendre transformation (7) is a Lie transformation.

Infinitesimal Lie transformations can be prolonged as regular Lie transformations. Namely let X be an infinitesimal Lie transformation of J^kE and let $\{\phi_t\}$ be its flow of Lie transformations. The first prolongation $\{\phi_t^{(1)}\}$ is a flow on $J^{k+1}E$ (a priori only defined in an open and dense subset). Additionally, as, for small t, ϕ_t is close to the identity, it follows that, for every $z' \in J^{k+1}E$ there is t such that $\phi_t^{(1)}$ is defined in z'. So the infinitesimal generator of $\{\phi_t^{(1)}\}$ is a well-defined vector field $X^{(1)}$ on the whole $J^{k+1}E$.

Definition 4.11. $X^{(1)}$ is the (first) *prolongation* of X.

We also put

$$X^{(r)} := X^{\underbrace{(1)(1)\cdots(1)}_{r \text{ times}}}$$

The following *Infinitesimal Lie-Bäcklund Theorem* is an easy consequence of Theorem 4.5.

Theorem 4.12 (Infinitesimal Lie-Bäcklund)**.**

1. *Let Y be an infinitesimal Lie transformation of J^kE, then $Y = X_1^{(k-1)}$ for some, necessarily unique, infinitesimal Lie transformation X_1 of J^1E. The correspondence $Y \mapsto X_1$ is one-to-one and Lie algebra property preserving.*
2. *In the additional hypothesis that $m \neq 1$, then, even more, $X = X^{(k)}$ for some point* infinitesimal point transformation $X \in \mathfrak{X}(E)$. *The correspondence $Y \mapsto X$ is one-to-one and Lie algebra property preserving.*

Exercise 4.13. Prove the Infinitesimal Lie-Bäcklund Theorem 4.12.

Example 4.14. We want to find coordinate formulas for the first prolongation of an infinitesimal point transformation. So let $X \in \mathfrak{X}(E)$ be locally given by

$$X = X^i \frac{\partial}{\partial x^i} + U^\alpha \frac{\partial}{\partial u^\alpha}, \tag{8}$$

$X^i = X^i(x, u)$, $U^\alpha = U^\alpha(x, u)$. As $X^{(1)}$, projects onto X, it is locally given by

$$X^{(1)} = X + U_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \tag{9}$$

$U_i^\alpha = U_i^\alpha(x, u, u')$, and we want to find the U_i^α in terms of the X^i and the U^α. In view of the infinitesimal Lie-Bäcklund Theorem, the U_i^α are completely determined by the condition that $X^{(1)}$ is an infinitesimal Lie transformation. From Exercise 3.11.(3) this is equivalent to

$$D_j \lrcorner \mathcal{L}_{X^{(1)}} \omega^\alpha = \frac{\partial}{\partial u_j^\beta} \lrcorner \mathcal{L}_{X^{(1)}} \omega^\alpha = 0.$$

Now, for every $Y \in \Gamma(C)$,

$$Y \lrcorner \mathcal{L}_{X^{(1)}} \omega^\alpha = [Y, X^{(1)}] \lrcorner \omega^\alpha.$$

Since the X^i and the U^α do only depend on (x, u), it easily follows that $\mathcal{L}_{X^{(1)}} \omega^\alpha$ annihilates the $\partial/\partial u_j^\beta$ identically. So it remains to check when does $i_{D_i} \mathcal{L}_{X^{(1)}} \omega^\alpha$ vanish. We have

$$D_j \lrcorner \mathcal{L}_{X^{(1)}} \omega^\alpha = [D_j, X^{(1)}] \lrcorner \omega^\alpha = D_j U^\alpha - u_i^\alpha D_j X^i - U_j^\alpha.$$

We conclude that (9) defines an infinitesimal Lie transformation iff

$$U_j^\alpha = D_j U^\alpha - u_i^\alpha D_j X^i, \tag{10}$$

which can be realized on the whole $J^1 E$.

Exercise 4.15. Let $X \in \mathfrak{X}(E)$ be an infinitesimal point transformation locally given by (8). Find coordinate formulas for the second prolongation $X^{(2)}$.

It follows from Theorem 4.12 that the Lie algebra of infinitesimal Lie transformations of $J^k E$ is actually isomorphic to

(1) the Lie algebra $\mathfrak{X}(E)$ if $m > 1$,
(2) the Lie algebra of infinitesimal Lie transformations of $J^1 E$ if $m = 1$.

In Section 7 we will study more closely the Lie algebra of infinitesimal Lie transformations of $J^1 E$, when $m = 1$.

5. Jets of submanifolds

In Differential Geometry, there are many examples of PDEs imposed on submanifolds of a given manifold: Lagrangian submanifolds, and minimal surfaces provide examples from symplectic and Riemannian geometry. For this reason it is important to develop a theory of "derivatives" or *jets of submanifolds*. In this short section we will outline this theory. As it is basically the same as the theory of jets of sections up to global, topological issues, we will go immediately back to the latter in the next section, leaving to the reader the necessary adaptations for submanifolds.

In this section E is an $n + m$-dimensional manifold (not necessarily a fiber bundle) and we look at n-dimensional submanifolds $N \subseteq E$. Let (x, u), $x = (x^1, \dots, x^n)$, $u = (u^1, \dots, u^m)$ be coordinates on E around a point of N such that, locally

$$N : u^\alpha = s^\alpha(x)$$

Definition 5.1. Coordinates (x, u) are said to be *adapted* to N.

Adapted coordinates do always exist by definition of submanifold. Now let N, O be two submanifolds such that $N \cap O \neq \varnothing$, and let $e_0 \in N \cap O$.

Exercise 5.2. Show that, around e_0, there are coordinates adapted to both N and O.

Proposition 5.3. *Let N, O be intersecting n-dimensional submanifolds of E and $e_0 \in N \cap O$. Let (x, u) be coordinates around e_0 adapted to both N, O, so that, locally*

$$N : u^\alpha = s^\alpha(x), \quad \textit{and} \quad O : u^\alpha = t^\alpha(x),$$

and let (x_0, u_0) be coordinates of e_0. Denote by $\mu_{N,e_0} \subseteq C^\infty(N)$ (resp. $\mu_{O,e_0} \subseteq C^\infty(O)$) the ideal consisting of functions h such that $h(e_0) = 0$. For every $k \geq 0$, the following two conditions are equivalent:

(1) *for all I such that $|I| \leq k$*

$$s^\alpha{}_{,I}(x_0) = t^\alpha{}_{,I}(x_0);$$

(2) *if $f \in C^\infty(E)$ then*

$$f|_N \in \mu^{k+1}_{N,e_0} \quad \textit{iff} \quad f|_O \in \mu^{k+1}_{O,e_0}.$$

Exercise 5.4. Prove Proposition 5.3 (*Hint: use Proposition 2.9*).

Definition 5.5. Two n-dimensional submanifolds N, O are *tangent up to order $k \geq 0$* at e_0 if $e_0 \in N \cap O$, and one of the equivalent conditions of Proposition 5.3 is satisfied. In this case we write $N \sim^k_{e_0} O$.

Remark 5.6.

N, O are tangent up to order 0 at e_0 if $e_0 \in N \cap O$.

N, O are tangent up to order 1 at e_0 if they are tangent up to order 0 and, additionally, they are tangent at e_0.

Tangency up to order k at e_0 is an equivalence relation on the set of all n-dimensional submanifold through e_0. The equivalence class of N is denoted $j^k_{e_0} N$ and called the *k-jet* of N at e_0.

Denote by $J^k_{e_0}(E, n)$ the space of all k-jets of n-dimensional submanifodls of E at e_0 and

$$J^k(E, n) := \bigsqcup_{e_0 \in E} J^k_{e_0}(E, n).$$

There are obvious surjections $p : J^k(E, n) \to J^{k-1}(E, n)$, $j^k_e N \mapsto j^{k-1}_e N$ which consist in "forgetting the last derivative".

$J^0(E, n)$ identifies with E, under $j^0_e N \mapsto e$. So $p \circ \cdots \circ p : J^k(E, n) \to J^0 E$ identifies with the *target map* $p_E : J^k(E, n) \to E$, $j^k_e N \mapsto e$.

Remark 5.7. We can put coordinates on $J^k(E, n)$ as follows. Let $j^k_{e_0} N_0$ be a point in $J^k(E, n)$, and let (x, u) be coordinates on E adapted to N, in a neighborhood U of e, and let $U_k \subseteq J^k(E, n)$ consist of submanifolds N of U such that (x, u) are also adapted to N. Finally, let $z = j^k_e N \in U_k$, with $N : u^\alpha = s^\alpha(x)$, and put

$$u^\alpha_I(z) = s^\alpha{}_{,I}(x), \quad \alpha = 1, \ldots, m, \quad I \text{ a multi-index s.t. } |I| \leq k.$$

Proposition 5.8. $(U_k, (x, \ldots, u_I, \ldots))$ *is a chart on* $J^k(E, n)$ *(called a* standard chart*). Any two such charts are compatible. With this atlas,* $J^k(E, n)$ *is a smooth manifold,*

$$\dim J^k(E, n) = n + m\binom{n+k}{k},$$

and both $p : J^k(E, n) \to J^{k-1}(E, n)$ *and* $p_E : J^k(E, n) \to E$ *are fiber bundles.*

Exercise 5.9. Prove Proposition 5.8. Prove that the fibers of $p : J^k(E, n) \to J^{k-1}(E, n)$ are diffeomorphic to some Euclidean space only when $k > 1$.

Remark 5.10. Fibers of $J^1(E, n) \to E$ are Grassmannians of n-dimensional subspaces in the tangent spaces to E.

Exercise 5.11. Let $E \to M$ be a fiber bundle, with $\dim M = n$. Prove that, for all k, there is a canonical embedding

$$J^k E \hookrightarrow J^k(E, n)$$

whose image is an open and dense submanifold.

Let N be an n-dimensional submanifold of E.

Definition 5.12. The *k-jet prolongation of N* is the following map:

$$j^k N : N \to J^k(E, n), \quad e \mapsto j^k_e N.$$

Exercise 5.13. Show that $j^k N$ is a smooth embedding.

The image of $j^k N$ is a submanifold of $j^k(E, n)$ (diffeomeorphic to N) denoted by $N^{(k)}$. A jet-plane in $J^k(E, n)$ is a tangent space to some $N^{(k)}$ at some point. Jet-planes of $J^k(E, n)$ span a maximally non-integrable distribution: the *Cartan distribution* on $J^k(E, n)$.

Exercise 5.14. Develop the theory of the Cartan distibution on $J^k(E, n)$ along the same lines as we did for $J^k E$.

6. PDEs

We finally discuss PDEs. A system of PDEs, or shortly a PDE, is a (finite) set of equations imposed on some functions (dependent variables) of some independent variables, and their derivatives, up to some finite order. So, it is natural to give the following, precise

Definition 6.1. A *system of k-th order PDEs*, or shortly a PDE, imposed on sections of a fiber bundle $\pi : E \to M$, is a submanifold $\mathcal{E} \subseteq J^k E$. A *solution* of $\mathcal{E}$ is then a local section s of E such that $j^k s$ takes values in $\mathcal{E}$.

Remark 6.2. One can define PDEs imposed on n-dimensional submanifolds of a given $(n + m)$-dimensional manifold E, simply replacing $J^k E$ with $J^k(E, n)$.

According to the definition, a PDE does locally look like

$$\mathcal{E} : F_a(x, \ldots, u_I, \ldots) = 0$$

for some local functions F_a on $J^k E$. A section s of E locally given by

$$s : u^\alpha = s^\alpha(x)$$

is then a solution iff, locally,

$$F_a(x, \ldots, s_{,I}(x), \ldots) = 0,$$

which is exactly what we expect from a PDE.

Given a PDE $\mathcal{E} \subseteq J^k E$, we consider the (generically non-regular distribution) on it:

$$C(\mathcal{E}) : z \mapsto C(\mathcal{E})_z := C_z \cap T_z \mathcal{E}.$$

Exercise 6.3. Prove that if the rank of $C(\mathcal{E})$ is constant, then $C(\mathcal{E})$ is a regular distribution (*Hint: the intersection of two vector subbundles is a vector subbundle iff its rank is constant.*).

Remark 6.4. We usually demand some minimal regularity conditions on $\mathcal{E}$ which make the theory reasonable and tractable at the same time. Namely, we assume that

(1) $\pi : \mathcal{E} \to M$ is surjective,
(2) $C(\mathcal{E})$ has constant rank (hence it is a regular distribution),
(3) $d\pi : C(\mathcal{E}) \to TM$ is (point-wise) surjective (hence $\pi : \mathcal{E} \to M$ is a submersion).

Condition (1) is a necessary condition for having solutions around every point of M. Condition (3) is a necessary condition for having a solution s through every point z of $\mathcal{E}$ (i.e. such that z belongs to the graph of $j^k s$). Suppose we have (3) but not (1). Then $\pi : \mathcal{E} \to M$ is a submersion, hence an open map, and we can replace M by $\pi(\mathcal{E})$ to force Condition (1). Condition (2) and (3) together make things smooth ruling out certain singularities. We will not comment further on (1)-(3).

Remark 6.5. A PDE can be often presented as the *zero locus of a differential operator*. Let $W \to M$ be a vector bundle. By definition, a (non-necessarily linear) W-valued k-th order *differential operator* (*DO*) on E is a bundle map $F : J^k E \to W$. Every DO determines a map Δ_F from sections of E to sections of W as follows:

$$\Delta_F(s) := F \circ j^k s.$$

The *zero locus* $Z(F)$ of F is the pre-image $F^{-1}(0_W)$ of the image 0_W of the zero section of $W \to M$, and one usually assumes regularity conditions on F that guarantee that $Z(F)$ is a PDE. Clearly, a section of E is a solution of $Z(F)$ iff $\Delta_F(s) = 0$. However F contains much more information than $Z(F)$.

Remark 6.6. When $E \to M$ is a vector bundle, then $J^k E \to M$ is a vector bundle as well (Exercise 2.16). In this situation, a *linear* PDE, is a PDE $\mathcal{E} \subseteq J^k E$, which is, additionally, a vector subbundle.

Remark 6.7. From Proposition 3.31, (locally) there is a one-to-one correspondence between solutions of $\mathcal{E}$ and π-horizontal, n-dimensional, integral submanifolds of $C(\mathcal{E})$. Sometimes it is useful to look at generic (not necessarily π-horizontal) n-dimensional, integral submanifolds. We call them *generalized solutions* and they play an important role in the theory of singularities of solutions.

Definition 6.8. A *Lie symmetry* of a PDE $\mathcal{E} \subseteq J^k E$ is a Lie transformation ϕ of $J^k E$ preserving $\mathcal{E}$, i.e. $\phi(\mathcal{E}) = \mathcal{E}$. An *infinitesimal Lie symmetry* is a vector field X on $J^k E$ generating a flow of Lie symmetries, i.e. X is an infinitesimal Lie transformation tangent to $\mathcal{E}$.

Exercise 6.9. Let $\mathcal{E}$ be a PDE, ϕ a Lie symmetry, X an infinitesimal Lie symmetry, and s a solution. Let $\{\phi_t\}$ be the flow of X. Prove that

(1) ϕ maps the graph of $j^k s$ to a (usually new) *generalized solution.*
(2) for sufficiently small t, ϕ_t maps the graph of $j^k s$ to the graph of $j^k s_t$ for some (usually new) solution s_t.

Remark 6.10. Finding Lie symmetries of a PDE $\mathcal{E}$ is usually complicated, because it involves solving *non-linear PDEs* as hard as (if not harder than) $\mathcal{E}$. On another hand, finding infinitesimal Lie symmetries is simpler, because it involves solving *linear PDEs*. So what one usually does is computing infinitesimal Lie symmetries and then integrating them to get (some) Lie symmetries. In what follows we concentrate on infinitesimal Lie symmetries.

Remark 6.11. In view of Exercise 6.9, knowing symmetries may help finding new solutions from given ones. One can also look for *symmetric solutions*. Namely, let $\mathcal{E}$ be a PDE and let X be an infinitesimal Lie symmetry. A solution s is *symmetric* under X if X is tangent to the graph of $j^k s$, in other words the flow of X leaves the graph of $j^k s$ unchanged. Finding symmetric solutions is often easier than finding generic solutions (see below).

We conclude this section discussing briefly how to find infinitesimal symmetries in practice. Begin with a

Lemma 6.12. *Let Y be an infinitesimal Lie transformation of $J^k E$, and let X_1 be its projection down to $J^1 E$ (it exists in view of the Infinitesimal Lie-Bäcklund Theorem). Then Y is completely determined by $\theta(X_1) \in \Gamma(p^* V)$, i.e. correspondence*

$$\mathfrak{X}_C \to \Gamma(p^* V), \quad Y \mapsto \theta(X_1) \tag{11}$$

is injective.

Proof. When $m > 1$, the claim follows easily from Example 4.14. In this case (see Example 10 for the notation),

$$X_1 = X^{(1)} = X^i \frac{\partial}{\partial x^i} + U^\alpha \frac{\partial}{\partial u^\alpha} + \left(D_j U^\alpha - u_i^\alpha D_j X^i\right) \frac{\partial}{\partial u_j^\alpha}$$

and

$$\theta(X_1) = \left(U^\alpha - u^\alpha_i X^i\right) \frac{\partial}{\partial u^\alpha}.$$

As X^i and U^α do only depend on (x,u) then the functions $\lambda^\alpha := U^\alpha - u^\alpha_i X^i$ completely determine X, hence Y.

The case $m = 1$ will be discussed in details in next section. $\square$

Remark 6.13. We will see in the next section that, for $m = 1$, correspondence (11) is also surjective.

Definition 6.14. Sections of p^*V in the image of (11) are called *generating sections* (of Lie transformations). The space of generating sections is denoted by $\varkappa_{\mathrm{cl}}$ and inherits from infinitesimal Lie transformations a Lie bracket, denoted $\{-,-\}$, and called the *Jacobi bracket*. The infinitesimal Lie transformation corresponding to the generating section λ, will be denoted by X_λ.

Exercise 6.15. Let λ be a generating section, let $\{\phi_t\}$ be the flow of X_λ and suppose that ϕ_t maps the graph of $j^k s$ to the graph of $j^k s_t$ (see Exercise 6.9). Show that s_t is then a solution of the following *evolutionary PDE*:

$$\frac{ds_t}{dt} = \lambda \circ j^1 s_t,$$

which in local coordinates looks like

$$\frac{ds^\alpha_t}{dt}(x) = \lambda^\alpha(x, s(x), s'(x)).$$

Conclude that s is symmetric under X_λ iff λ vanishes on the graph of $j^1 s$.

Now, let $\mathcal{E}$ be a PDE locally given by

$$\mathcal{E} : F_a(x, \ldots, u_I, \ldots)$$

As usual, we assume that $F = (F_1, \ldots, F_p) : J^k E \to \mathbb{R}^p$ is a submersion around $\mathcal{E}$. How do we find infinitesimal Lie symmetries of $\mathcal{E}$? We start with a generating section λ and compute X_λ (this involves prolonging an infinitesimal Lie transformation several times and could be computationally rather hard. For this reason one often uses computer algebra). Second we impose the tangency condition

$$X_\lambda(F_a)|_{\mathcal{E}} = 0. \tag{12}$$

This is a (usually not too hard) system of linear PDEs in λ.

Definition 6.16. Equation (12) is called the *defining equation of symmetries*. Solutions of (12) are *generating sections of symmetries*.

Exercise 6.17. Compute *point symmetries*, i.e. Lie symmetries of the form $X^{(2)}$ for some infinitesimal point transformation X, of the *Burger's equation*:

$$u_t = u_{xx} + u u_x.$$

Remark 6.18. Let $\mathcal{E}$ be a PDE and let λ be the generating section of a symmetry of $\mathcal{E}$. One can look for symmetric solutions (under X_λ), solving $\mathcal{E}$ coupled to $\lambda = 0$. This is often an overdetermined system much easier to solve than its subsystem $\mathcal{E}$.

Remark 6.19. There is a more intrinsic definition of a(n infinitesimal) symmetry of a PDE $\mathcal{E}$. By intrinsic, we mean that it is independent of the *extrinsic* geometry encoded in the embedding $\mathcal{E} \hookrightarrow J^kE$ and does only depend on the *intrinsic* geometry of $\mathcal{E}$ and its distribution $C(\mathcal{E})$. Specifically, an *intrinsic infinitesimal symmetry* of $\mathcal{E}$, is an infinitesimal symmetry of the distribution $C(\mathcal{E})$. Notice that, if X is a Lie symmetry of $\mathcal{E}$, then $X|_{\mathcal{E}}$ is tangent to $\mathcal{E}$ and it is actually an intrinsic infinitesimal symmetry. But, in general, not all intrinsic infinitesimal symmetries arise in this way. However, for a large, but reasonable, class of PDEs $\mathcal{E}$, the intrinsic data $(\mathcal{E}, C(\mathcal{E}))$ determine the extrinsic geometry completely, and all intrinsic symmetries come from extrinsic ones. So studying extrinsic symmetries only is not too much restrictive.

7. Contact Geometry: first order PDEs in 1 dependent variable

As we already mentioned, the case $k = m = 1$ of first order PDEs in one dependent variable is peculiar. We treat it separately in this section. First of all, notice that, in this case

$$\dim J^kE = 2n + 1 \quad \text{and} \quad \operatorname{rank} C = 2n.$$

In particular, the Cartan distribution is a maximally non-integrable hyperplane distribution, and all locally maximal integral submanifolds share the same dimension n. More generally we give the following

Definition 7.1. A *contact distribution*, or a *contact structure*, is a maximally non-integrable hyperplane distribution. A *contact manifold* (N, C) is a manifold N equipped with a contact structure C. A *Legendrian submanifold* of (N, C) is a locally maximal integral submanifold of C. Two contact manifolds (N_1, C_1) and (N_2, C_2) are *contactomorphic* if there is a *contactomorphism* $\phi : (N_1, C_1) \to (N_2, C_2)$, i.e. a diffeomorphism $\phi : N_1 \to N_2$ preserving the contact structures: $d\phi(C_1) = C_2$. An *infinitesimal contactomorphism*, or a *contact vector field*, of (N, C) is a vector field $X \in \mathfrak{X}(N)$ which generates a flow of contactomorphisms.

So, in the case $m = 1$, J^1E is a contact manifold, graphs of jet prolongations are Legendrian submanifolds, and Lie transformations (resp. infinitesimal Lie transformations) are contactomorphisms (resp. infinitesimal contactomorphisms). Actually, the theory of first oder PDEs in one dependent variable can be developed on a generic contact manifold (N, C), and this is what we do here. We start with a local structure theorem that shows that, locally, all contact manifolds of the same dimension look the same, in particular, they look like a 1-jet space of a rank one fiber bundle.

Lemma 7.2 (Darboux)**.** *Let (N, C) be a contact manifold. Then* $\dim N = 2n + 1$ *for some positive integer n, and locally there are coordinates (x, u, u') (called* Darboux coordinates*) such that*

$$C = \left\langle \dots, D_i, \dots, \frac{\partial}{\partial u_i}, \dots \right\rangle,$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u}, \quad i = 1, \dots, n.$$

Proof. We use the *Darboux Lemma* from *symplectic geometry* which states that, if Ω is a non-degenerate and closed 2-form, then locally there are coordinates (x, u') such that

$$\Omega = dx^i \wedge du_i.$$

Now, locally C is the kernel of a 1-form θ, and C being maximally non-integrable implies that $d\theta|_C$ is non-degenerate (Exercise 7.3). So $\operatorname{rank} C$ is even, and $\dim N$ is odd. The kernel of $d\theta$, i.e. the distribution K spanned by tangent vectors v such that

$$v \lrcorner\, d\theta = 0 \tag{13}$$

is necessarily rank 1. It follows from the Frobenius Theorem, that, locally, K is the vertical distribution of a rank one fiber bundle $\rho : N \to N_0$. From (13) and $dd\theta = 0$ it follows that $d\theta$ is a ρ-basic form, i.e. $d\theta = \rho^*(\Omega)$ for a, necessarily unique, non-degenerate, and closed 2-form Ω on N_0. Choose Darboux coordinates (x, u') on (N_0, Ω), i.e. locally

$$\Omega = dx^i \wedge du_i.$$

The pull-back $\rho^*(x^i)$, $\rho^*(u_i)$ are functions on N that we denote by x^i, u_i again. So

$$d\theta = dx^i \wedge du_i$$

and, from the Poincaré Lemma

$$\theta = du - u_i dx^i$$

for some local function u on N. It remains to prove that u is functionally independent of (x, u'). This immediately follows from Exercise 7.3. □

Exercise 7.3. Let N be a manifold and let C be an hyperplane distribution on N which can be presented as the kernel of a global 1-form θ. Prove that the following conditions are equivalent:

(1) C is maximally non-integrable;
(2) $d\theta|_C$ is non-degenerate;
(3) $\dim N = 2n + 1$, for some n, and $\theta \wedge (d\theta)^n \neq 0$ everywhere on N.

Corollary 7.4. *Two contact manifolds are locally contactomorphic iff they have the same dimension.*

Corollary 7.5. *Legendrian submanifolds of a $(2n + 1)$-dimensional contact manifold are n-dimensional.*

Exercise 7.6. Let (N, C) be a contact manifold, let $\mathcal{L} \subseteq N$ be a Legendrian submanifold, and let $z \in L$. Prove that there are Darboux coordinates (x, u, u') around z such that L is π-horizontal, were π is the projection $(x, u, u') \mapsto x$. Conclude that there are Darboux coordinates (x, u) such that

$$\mathcal{L} : \begin{cases} u = f(x) \\ u_i = \frac{\partial f}{\partial x^i}(x) \end{cases},$$

for some function $f = f(x)$.

We now describe infinitesimal symmetries of a contact manifold (N, C), i.e. contact vector fields. First of all, denote by $L := TM/C$ the quotient vector bundle. As C is hyperplane, L is a *line bundle*, i.e. a rank one vector bundle. Let $\theta \in \Omega^1(N, L)$ be the structure form of C.

Proposition 7.7. *The short exact sequence of vector spaces*

$$0 \to \Gamma(C) \to \mathfrak{X}(N) \xrightarrow{\theta} \Gamma(L) \to 0 \tag{14}$$

splits in a canonical way. The image of the splitting $\Gamma(L) \to \mathfrak{X}(N)$ *consists of contact vector fields. In particular, there is a canonical isomorphism of vector spaces*

$$\Gamma(L) \to \mathfrak{X}_C.$$

Proof. Consider the curvature $\omega : \wedge^2 C \to L$ of C. By contraction, it induces a vector bundle morphism $\omega_\flat : C \to C^* \otimes L$. As C is maximally non-integrable, ω is non-degenerate, hence $\omega_\flat$ is invertible. Denote by $\omega^\sharp : C^* \otimes L \to C$ its inverse. Now, let X be any vector field on N. Consider the map

$$\varphi_X : \Gamma(C) \to \Gamma(L), \quad Y \mapsto \theta\left([X, Y]\right). \tag{15}$$

It is easy to see that φ_X is $C^\infty(N)$-linear (Exercise 7.8.(1)), hence it comes from a vector bundle morphism $C \to L$, or, which is the same, a section of $C^* \otimes L$, that we denote by φ_X again. Composing with $\omega^\sharp$ we get a map

$$p_C : \mathfrak{X}(N) \to \Gamma(C), \quad X \mapsto \omega^\sharp(\varphi_X). \tag{16}$$

A straightforward computation shows that p_C splits the sequence (14) (Exercise 7.8.(2)).

It remains to show that $\ker p_C = \mathfrak{X}_C$. This immediately follows from the definition (Exercise 7.8.(3)). □

Exercise 7.8. Prove that

(1) the map φ_X in (15) is $C^\infty(N)$-linear;
(2) the map p_C in (16) splits (14);
(3) the kernel of p_C is $\mathfrak{X}_C$.

We denote

$$\Gamma(L) \to \mathfrak{X}_C, \quad \lambda \mapsto X_\lambda$$

the vector space isomorphism of Proposition 7.7. We can use it to transfer the Lie algebra structure from $\mathfrak{X}_C$ to $\Gamma(L)$. Accordingly, $\Gamma(L)$ is canonically equipped with a Lie bracket denoted

$$\{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L).$$

Definition 7.9. The line bundle L, together with the Lie bracket $\{-,-\}$ on its sections, is called *the Jacobi bundle* of (N, C), $\{-,-\}$ is the *Jacobi bracket*, and every section of L is a *generating section* of a contactomorphism. The vector field X_λ, corresponding to the generating section λ, is called the *Hamiltonian vector field* corresponding to λ.

Exercise 7.10. Prove that

$$\{\lambda, f\mu\} = X_\lambda(f)\mu + f\{\lambda, \mu\},$$

for all $\lambda, \mu \in \Gamma(L)$, and all $f \in C^\infty(N)$.

Exercise 7.11. Let (N, C) be a contact manifold and let (x, u, u') be Darboux coordinates. After noticing that $\partial/\partial u$ is a contact vector field, put $I = \theta(\partial/\partial u)$ so that $\partial/\partial u = X_I$, and I generates $\Gamma(L)$ locally. Prove that the Jacobi bracket is locally given by

$$\{f \cdot I, g \cdot I\} = \left(D_i f \frac{\partial g}{\partial u_i} - D_i g \frac{\partial f}{\partial u_i} + f \frac{\partial g}{\partial u} - g \frac{\partial f}{\partial u} \right) \cdot I,$$

and the Hamiltonian vector field corresponding to $f \cdot I$ is

$$X_{f \cdot I} = D_i f \frac{\partial}{\partial u_i} - \frac{\partial f}{\partial u_i} D_i + f \frac{\partial}{\partial u},$$

for all $f, g \in C^\infty(N)$.

7.1. **The method of characteristics.**

Definition 7.12. A (single) *first order PDE in one dependent variable*, in contact geometry, is an hypersurface $\mathcal{E}$ in a $(2n+1)$-dimensional contact manifold (N, C) such that $C(\mathcal{E}) := C \cap T\mathcal{E}$ has constant rank (hence, it is a regular distribution on $\mathcal{E}$). A *solution* of $\mathcal{E}$ is a Legendrian submanifold $\mathcal{L}$ of (N, C) such that $\mathcal{L} \subseteq \mathcal{E}$, equivalently, it is an n-dimensional integral submanifold of $C(\mathcal{E})$.

Let $\mathcal{E}$ be a first order PDE in a $(2n+1)$-dimensional contact manifold (N, C). The classical method of characteristics has a simple geometric interpretation within contact geometry. To see this first notice that $\operatorname{rank} C(\mathcal{E}) = 2n - 1$. Hence the curvature ω, when restricted to $C(\mathcal{E})$, degenerates along a rank 1, hence integrable, distribution $K(\mathcal{E}) \subseteq C(\mathcal{E})$. According to Frobenius Theorem, $\mathcal{E}$ is then *foliated* by integral curves of $K(\mathcal{E})$. The distribution $K(\mathcal{E})$ is the *characteristic distribution* of $\mathcal{E}$, and its integral foliation is the *characteristic foliation* of $\mathcal{E}$. Finally, leaves of $K(\mathcal{E})$ are *characteristic curves.*

Proposition 7.13. *Let $\omega_\mathcal{E}$ be the curvature of the distribution $C(\mathcal{E})$. Then $\omega_\mathcal{E} = \omega|_{C(\mathcal{E})}$, hence $K(\mathcal{E})$ is the (constant rank) kernel of $\omega_\mathcal{E}$, and it is spanned by characteristic symmetries of $C(\mathcal{E})$.*

Proposition 7.14. *Denote by $\Gamma_{\mathcal{E}} \subseteq \Gamma(L)$ the submodule consisting of generating sections λ such that $\lambda|_{\mathcal{E}} = 0$. Then*

(1) *If $\lambda \in \Gamma_{\mathcal{E}}$, then X_λ is tangent to $\mathcal{E}$, and $X_\lambda|_{\mathcal{E}}$ is a characteristic symmetry of $C(\mathcal{E})$.*
(2) *$K(\mathcal{E})$ si spanned by the X_λ with $\lambda \in \Gamma_{\mathcal{E}}$.*
(3) *If $\Gamma_{\mathcal{E}}$ has exactly one generator λ, then $X_\lambda|_z \neq 0$ for all $z \in \mathcal{E}$, and $K(\mathcal{E})$ is spanned by X_λ.*

Exercise 7.15. Prove Proposition 7.14.

Definition 7.16. A set of *non-characteristic initial data* for $\mathcal{E}$ is an $(n-1)$-dimensional integral submanifold Σ of $C(\mathcal{E})$ such that $T_z\Sigma \cap K(\mathcal{E})_z = 0$ for all $z \in \Sigma$.

Proposition 7.17. *Let Σ be a set of non-characteristic initial data for $\mathcal{E}$. Then, locally, around every point of Σ, there are Darboux coordinates (x, u, u') such that*

(1) *$\mathcal{E}$ is locally given by*
$$\mathcal{E} : u_n = G(x^1, \dots, x^n, u, u_1, \dots, u_{n-1})$$
(2) *Σ is locally given by*
$$\Sigma : \begin{cases} x^n = 0 \\ u = f_0(x^1, \dots, x^{n-1}) \\ u_i = \dfrac{\partial f_0}{\partial x^i}(x^1, \dots, x^{n-1}) & \text{for } i = 1, \dots, n-1 \\ u_n = G\left(x^1, \dots, x^{n-1}, 0, f_0, \dots, \dfrac{\partial f_0}{\partial x^i}, \dots\right). \end{cases}$$

In particular, Σ is completely determined by the function $f_0 = f_0(x^1, \dots, x^n)$.

Remark 7.18. Proposition 7.17 motivates the terminology "set of initial data" used for Σ.

Exercise 7.19. Prove Proposition 7.17.

Proposition 7.20 (The Method of Characteristics)**.** *Let $\mathcal{E}$ be a first order PDE in one dependent variable, let $\Sigma \subseteq \mathcal{E}$ be a set of non-characteristic initial data of $\mathcal{E}$, and let $\lambda \in \Gamma_{\mathcal{E}}$ be such that X_λ generates $K(\mathcal{E})$ at least around Σ. Then the flow-out of Σ along X_λ is a solution of $\mathcal{E}$ (independent of λ).*

Exercise 7.21. Prove Proposition 7.20.

Example 7.22. Consider the following first order PDE in 1 dependent variable u, and 2 independent variables x, t:
$$u = u_x u_t.$$
We want to solve it with initial data:
$$u|_{t=0} = x^2. \tag{17}$$

To do this we work in the contact manifold $\mathbb{R}^5$ with Darboux coordinates (x, t, u, u_x, u_t). Then

$$\mathcal{E} : F(x, t, u, u_x, u_t) := u - u_x u_t = 0,$$

and, when $u_x \neq 0$, the initial data (17) are encoded by the following integral curve of $C(\mathcal{E})$:

$$\Sigma : \begin{cases} t = 0 \\ u = x^2 \\ u_x = 2x \\ u_t = u/u_x = x/2. \end{cases}$$

If we use (x, t, u_x, u_t) to parameterize $\mathcal{E}$, then in such internal coordinates

$$\Sigma : \begin{cases} t = 0 \\ u_x = 2x \\ u_t = x/2. \end{cases}$$

and the characteristic distribution $K(\mathcal{E})$ is spanned by the Hamiltonian vector field

$$Y := X_{F \cdot I}|_{\mathcal{E}} = u_x \frac{\partial}{\partial u_x} + u_t \frac{\partial}{\partial u_t} + u_t \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial t},$$

whose integral curves $\gamma = \gamma(\varepsilon)$ are solutions of the following system of ODEs

$$\begin{cases} \dot{x} = u_t \\ \dot{t} = u_x \\ \dot{u}_x = u_x \\ \dot{u}_t = u_t \end{cases}$$

where a dot "$\dot{(-)}$" denotes a derivative wrt ε. The flow of Y is then

$$\begin{cases} x(\varepsilon) = x(0) + u_t(0)\,(e^\varepsilon - 1) \\ t(\varepsilon) = t(0) + u_x(0)\,(e^\varepsilon - 1) \\ u_x(\varepsilon) = u_x(0) e^\varepsilon \\ u_t(\varepsilon) = u_t(0) e^\varepsilon \end{cases}$$

Applying this to a point $(x(0), t(0), u_x(0), u_t(0)) = (x(0), 0, 2x(0), x(0)/2)$ of Σ we get the following parametric expression for the flow-out of Σ

$$\cup_\varepsilon \Sigma_\varepsilon : \begin{cases} t = 2x(0)\,(e^\varepsilon - 1) \\ x = \dfrac{x(0)}{4}(e^\varepsilon - 1) \\ u_x = 2x(0) e^\varepsilon \\ u_t = \dfrac{x(0)}{2} e^\varepsilon. \end{cases}$$

Eliminating the parameters $\varepsilon, x(0)$ we get

$$\cup_\varepsilon \Sigma_\varepsilon : \begin{cases} u_x = \dfrac{4x+t}{2} \\ u_t = \dfrac{4x+t}{8}. \end{cases}$$

Hence, the solution we are looking for is the solution

$$u = \frac{(4x+t)^2}{16}.$$

8. ∞-Jets

Given a PDE it is often useful, in order to check its integrability, to consider its total derivatives. In doing this, higher derivatives come into play. So it is convenient considering all derivatives "at the same time". This can be done via ∞-jets.

Let $E \to M$ be a fiber bundle.

Definition 8.1. Two local sections s, t of E defined around $x_0 \in M$ are *tangent up to order* ∞ at x_0 if $s \sim_{x_0}^k t$, for all $k \geq 0$. In this case we write $s \sim_{x_0}^\infty t$.

Tangency up to order ∞ at x_0 is an equivalence relation on the set of all local sections defined around x_0. The equivalence class of s is denoted $j_{x_0}^\infty s$ and called the ∞*-jet* of s at x_0. In practice it contains a full information on "all derivatives of s" at x_0.

Denote by $J_{x_0}^\infty E$ the space of all ∞-jets of sections of E at x_0 and

$$J^\infty E := \bigsqcup_{x_0 \in M} J_{x_0}^\infty E.$$

$J^\infty E$ is the ∞*-jet space*, or ∞*-jet bundle*, of E. In practice, points in $J^\infty E$ are Taylor series of sections of E at all possible points of M.

There are obvious maps $p_k : J^\infty E \to J^k E$, $j_x^\infty s \mapsto j_x^k s$ which consist in "forgetting highest derivative".

There is also a surjection, the *source map*, $\pi : J^\infty E \to M$, $j_x^\infty s \mapsto x$.

Proposition 8.2. *$J^\infty E$, together with maps p_k is an inverse limit of the sequence of surjections*

$$\cdots \xleftarrow{p} J^k E \xleftarrow{p} J^{k+1} E \longleftarrow \cdots .$$

In particular, the p_k are surjective.

Exercise 8.3. Prove Proposition 8.2 (*Hint: The proposition is actually equivalent to Borel Lemma*).

Remark 8.4. According to Proposition 8.2 points of $J^\infty E$ can be seen as *threads* in $\prod_k J^k E$, i.e. sequences $\{z_k\}$ such that $z_k \in J^k E$ and $p(z_k) = z_{k-1}$ for all k. If $z = j_x^\infty s$, then $z_k = j_x^k s$.

We can give to $J^\infty E$ the inverse limit topology, i.e. the coarsest topology such that all p_k are continuous. In this way $J^\infty E$ becomes a topological subspace in the product $\prod_k J^k E$. Clearly, it's not a finite dimensional manifold. However, there is a way (actually more than one) to make geometry on it, as we will show below.

With its inverse limit structure $J^\infty E$ is a *profinite dimensional manifold*.

8.1. **Profinite dimensional manifolds.**

Let

$$\cdots \xleftarrow{p} N_k \xleftarrow{p} N_{k+1} \longleftarrow \cdots , \tag{18}$$

be a sequence of surjective submersions, and let $(N_\infty, \{p_k\})$ be its inverse limit.

Definition 8.5. A pair $(N_\infty, \{p_k\})$ is a *profinite dimensional manifold*. A *smooth map* of profinite dimensional manifolds $(N_\infty, \{p_k\}) \to (O_\infty, \{q_k\})$ is a map $F : N_\infty \to O_\infty$ such that

$$F = \varprojlim F_k,$$

where $F_k : N_k \to O_{k+l}$ are smooth maps, only defined from some k on, such that $F_{k-1} \circ p = p \circ F_k$, with l some constant integer, i.e., if $\{z_k\}$ is a thread in N_∞, then

$$F(\{z_k\}) = \{F_k(z_k)\}.$$

Remark 8.6. Any smooth map of profinite dimensional manifolds is, in particular, continuous.

Remark 8.7. Profinite dimensional manifolds, together with smooth maps, form a category. Isomorphisms in this category are *diffeomorphisms*.

Example 8.8. Let $N_k = N$ and $p = \mathrm{id}$ for all k, then N_∞ is a profinite dimensional manifold. Topologically, it is homeomorphic to N. This construction embeds standard manifolds into profinite dimensional ones as a full subcategory.

Example 8.9. The ∞-jet space is a profinite dimensional manifold.

Example 8.10. Let $\boldsymbol{n} = \{n_k\}_k \subseteq \mathbb{N}_0$ be a non-decreasing sequence, and let

$$\cdots \xleftarrow{p} \mathbb{R}^{n_k} \xleftarrow{p} \mathbb{R}^{n_{k+1}} \longleftarrow \cdots,$$

be the sequence of projections forgetting the last coordinates. Its inverse limit $\mathbb{R}^{\boldsymbol{n}}$ is a profinite dimensional manifold.

Remark 8.11. A profinite dimensional manifolds N_∞ cannot be coordinatized in general, in the sense that N_∞ is not covered by open subsets homeomorphic to some $\mathbb{R}^{\boldsymbol{n}}$. However, sometimes, N_∞ *can* be coordinatized. This is the case, for instance, when $N_\infty = J^\infty E$.

We can put "coordinates" on $J^\infty E$ as follows. Let (x, u) be bundle coordinates on E, defined in a neighborhood U, and let $U_\infty := p_0^{-1}(U) \subseteq J^\infty E$. Clearly, U_∞ is an open subset. Let $z = j^\infty_x s \in U_\infty$, with $s : u^\alpha = s^\alpha(x)$, and put

$$u^\alpha_I(z) = s^\alpha{}_{,I}(x), \quad \alpha = 1, \ldots, m, \quad I \text{ a multi-index s.t. } |I| \geq 0.$$

Proposition 8.12. *$(U_\infty, (x, \ldots, u_I, \ldots))$ is a chart on $J^\infty E$ (called a* standard chart*), in the sense that $(x, \ldots, u_I, \ldots) : U_\infty \to \mathbb{R}^{\boldsymbol{N}}$ is a diffeomorphism (of profinite dimensional manifolds). Here $\boldsymbol{N} := \{N_k\}_k$ is the sequence defined by*

$$N_k = \dim J^k E = n + m\binom{n+k}{k}.$$

Exercise 8.13. Prove Proposition 8.12 (*Hint: First notice that U_∞ is a profinite dimensional manifold in an obvious way and that the inclusion $U_\infty \hookrightarrow J^\infty E$ is a smooth map*).

One can define differential calculus on profinite dimensional manifolds algebraically as follows. First notice that to sequence (18) it corresponds a sequence of algebra inclusions

$$\cdots \hookrightarrow C^\infty(N_k) \hookrightarrow C^\infty(N_{k+1}) \hookrightarrow \cdots , \tag{19}$$

here arrows are pull-backs along p. The *direct limit* of (19) is a *filtered algebra* denoted $C^\infty(N_\infty)$ and called the *algebra of smooth* (*local*) *functions on* N_∞. In practice a smooth function on N_∞ is a smooth function on N_k for some k.

Remark 8.14. Any smooth function $f \in C^\infty(N_\infty)$ can be seen as a map $f : N_\infty \to \mathbb{R}$ in an obvious way.

Exercise 8.15. Prove that

(1) any smooth function $f \in C^\infty(N_\infty)$ is a smooth map $f : N_\infty \to \mathbb{R}$;
(2) the pull-back $F^*(f)$ of a smooth function $f \in C^\infty(O_\infty)$ along a smooth map $F : N_\infty \to O_\infty$ is a smooth function.

Definition 8.16. A tangent vector to N_∞ at a point $z \in N_\infty$ is an $\mathbb{R}$-valued derivation of the algebra $C^\infty(N_\infty)$, i.e. and $\mathbb{R}$-linear map $v : C^\infty(N_\infty) \to \mathbb{R}$ satisfying the (pointed) *Leibniz rule*:

$$v(fg) = v(f)g(z) + f(z)v(g),$$

for all $f, g \in C^\infty(N_\infty)$

Remark 8.17. It immediately follows from the definition that tangent vectors to N_∞ at z form a vector space, denoted T_zN_∞ and called the *tangent space* to N_∞ at z.

Disjoint union

$$TN := \bigsqcup_{z \in N_\infty} T_zN_\infty$$

together with the obvious projection $\tau : TN_\infty \to N_\infty$, is called the *tangent bundle to* N_∞. It can be given the structure of a profinite dimensional manifold, so that τ is a smooth map, as follows. Let $v \in T_zN_\infty$. Restricting v to $C^\infty(N_k) \hookrightarrow C^\infty(N_\infty)$ defines a tangent vector to N_k at $\pi_k(z)$. So we have a map

$$dp_k : TN_\infty \to TN_k.$$

Proposition 8.18. *$(TN_\infty, \{dp_k\})$ is an inverse limit of*

$$\cdots \xleftarrow{dp} TN_k \xleftarrow{dp} TN_{k+1} \longleftarrow \cdots .$$

In particular, TN_∞ is a profinite dimensional manifold. Projection $\tau : TN_\infty \to N_\infty$ is smooth.

Exercise 8.19. Prove Proposition 8.18.

Exercise 8.20. Define the tangent map $dF : TN_\infty \to TO_\infty$ to a smooth map $F : N_\infty \to O_\infty$ of profinite dimensional manifolds, and show that this construction promote $N_\infty \mapsto TN_\infty$ to an endo-functor of the category of profinite dimensional manifolds.

Definition 8.21. A *vector field* on N_∞ is a derivation $X : C^\infty(N_\infty) \to C^\infty(N_\infty)$ such that

$$X(C^\infty(N_k)) \subseteq C^\infty(N_{k+l})$$

for all k, with l some constant integer.

Proposition 8.22. *The space of vector fields on N_∞ is a $C^\infty(N_\infty)$-module and a Lie algebra, denoted $\mathfrak{X}(N_\infty)$, under the commutator. Additionally*

$$[X, fY] = X(f)Y + f[X, Y],$$

for all $X, Y \in \mathfrak{X}(N_\infty)$, and all $f, g \in C^\infty(N_\infty)$.

Exercise 8.23. Prove Proposition 8.22.

Let $X \in \mathfrak{X}(N_\infty)$. For any $z \in N_\infty$ and any $f \in C^\infty(N_\infty)$, put

$$X_z(f) := X(f)(z).$$

It is immediate to see that $X_z : C^\infty(N_\infty) \to \mathbb{R}$ is a tangent vector at z.

Proposition 8.24. *Denote by $\Gamma(TN_\infty)$ the space of sections of TN_∞, i.e. smooth maps σ s.t. $\tau \circ \sigma = \mathrm{id}_{N_\infty}$.*

(1) *$\Gamma(TN_\infty)$, equipped with point-wise addition and multiplication, is a $C^\infty(N_\infty)$-module.*
(2) *Correspondence*

$$X \mapsto (z \mapsto X_z)$$

is a $C^\infty(N_\infty)$-module isomorphism between $\mathfrak{X}(N_\infty)$ and $\Gamma(TN_\infty)$.

Exercise 8.25. Prove Proposition 8.24.

Remark 8.26. A consequence of Proposition 8.24 is that vector fields can be "restricted" to open submanifolds and that a vector field is completely determined by its "restrictions" to open submanifolds covering N_∞.

Example 8.27 (Vector field on $J^\infty E$)**.** Let $E \to M$ be a fiber bundle, let $z \in J^\infty E$, and let v be a tangent vector to $J^\infty E$ at z. Put $z_k = p_k(z)$. Then z identifies with a thread $\{v_k\}$, with v_k a tangent vector to $J^k E$ at z_k such that $dp(v_k) = v_{k-1}$. If $(x, \ldots, u_I, \ldots)$ are standard coordinates around z, then v_k is of the form

$$v_k = v^i \left.\frac{\partial}{\partial x^i}\right|_{z_k} + \sum_{|I| \leq k} v_I^\alpha \left.\frac{\partial}{\partial u_I^\alpha}\right|_{z_k}$$

and v identifies with the formal series

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_{z} + \sum_{|I| \geq 0} v_I^\alpha \left.\frac{\partial}{\partial u_I^\alpha}\right|_{z}.$$

Here $v^i = v(x^i)$, and $v_I^\alpha = v(u_I^\alpha)$.

Similarly, let X be a vector field on $J^\infty E$. Restricting X to charts U_∞ we get vector fields on them. Additionally, X is completely determined by those restrictions. Any such restriction can be written as a formal series

$$X = X^i \frac{\partial}{\partial x^i} + \sum_{I \geq 0} X^\alpha_I \frac{\partial}{\partial u^\alpha_I},$$

where $X^i = X(x^i)$ and $X^\alpha_I = X(u^\alpha_I)$. In particular, for $|I| = k$, we have $X^\alpha_I \in C^\infty(J^{k+l}E)$, where l is the shift of X.

Exercise 8.28. Prove all the claims in Example 8.27.

We conclude this short introduction to profinite dimensional manifolds discussing *differential forms* on them.

Let N_∞ be a profinite dimensional manifold. Besides (19) there is a sequence of graded algebra inclusions

$$\cdots \hookrightarrow \Omega(N_k) \hookrightarrow \Omega(N_{k+1}) \hookrightarrow \cdots , \tag{20}$$

where the arrows are pull-backs along p. The direct limit of (20) is a *filtered, graded algebra* denoted $\Omega(N_\infty)$ and called the *algebra of differential forms on N_∞*. We denote by $\Omega^q(N_\infty)$ the q-th homogeneous piece of $\Omega(N_\infty)$.

Now, let X be a vector field, and let ω be a differential 1-form on N_∞. So $\omega \in \Omega^1(N_k)$ for some N_k. On another hand $X(C^\infty(N_k)) \subseteq C^\infty(N_{k+l})$ where l is the shift of X. In other words,

$$X_k := X|_{C^\infty(N_k)}$$

is a vector field, *relative to the projection $p \circ \cdots \circ p : N_{k+l} \to N_k$*. We can cotract it with ω to get a smooth function $i_X\omega$ on N_{k+l}. Regard $\omega(X) := i_X\omega$ as a smooth function on N_∞.

Proposition 8.29.

1. *$i_X : \Omega^1(N_\infty) \to C^\infty(N_\infty)$ is a $C^\infty(N_\infty)$-linear map.*
2. *Correspondence $X \mapsto i_X$ is a $C^\infty(N_\infty)$-module isomorphism between vector fields on N_∞ and* filtered *$C^\infty(N_\infty)$-linear maps $\Omega^1(N_\infty) \to C^\infty(N_\infty)$, i.e. linear maps $\phi : \Omega^1(N_\infty) \to C^\infty(N_\infty)$ such that $\phi(\Omega^1(N_k)) \subseteq C^\infty(N_{k+l})$ for some constant l.*

Exercise 8.30. Prove Proposition 8.29.

Exercise 8.31 (Filtered biduality)**.** Show that the obvious injection of $\Omega^1(N_\infty)$ into $C^\infty(N_\infty)$-linear maps $\mathfrak{X}(N_\infty) \to C^\infty(N_\infty)$ is one-to-one onto *filtered* linear maps (where $\mathfrak{X}(N_\infty)$ is filtered by the shift of vector fields).

Exercise 8.32. Define *Cartan calculus* on a profinite dimensional manifold.

8.2. **The Cartan distribution on ∞-jets.** We begin defining finite rank distributions on a profinite dimensional manifold. So, let N_∞ be a profinite dimensional manifold.

Definition 8.33. A rank k *distribution* D on N_∞ is a submodule of $\mathfrak{X}(N_\infty)$, denoted $\Gamma(D)$, such that, for every $z \in N_\infty$, the values Y_z, with $Y \in \Gamma(D)$, span a k-dimensional subspace in $T_z N$, denoted D_z. If $\Gamma(D)$ is locally spanned by independent vector field $Y_1, \ldots, Y_k$ we say that $Y_1, \ldots, Y_k$ span D (locally), and write

$$D = \langle Y_1, \ldots, Y_k \rangle.$$

Distribution D is *involutive* if, additionally, $\Gamma(D) \subseteq \mathfrak{X}(N_\infty)$ is a Lie subalgebra.

Remark 8.34. Notice that, if N_∞ is finite dimensional, then the above definition agrees with the usual one.

Let $E \to M$ be a fiber bundle, with $\dim M = n$. Then $J^\infty E$ is equipped with a canonical rank n involutive distribution C called the *Cartan distribution*. To see this, first notice that we can use the ∞-jet space to encode *derivative functions*. Let s be a section of E.

Definition 8.35. The *∞-jet prolongation of* s is the following smooth section of $J^\infty E \to M$:

$$j^\infty s : M \to J^\infty E, \quad x \mapsto j^\infty_x s.$$

Exercise 8.36. Show that $j^\infty s$ is actually a smooth section (*Hint: show that if $s : u^\alpha = s^\alpha(x)$ then*

$$j^\infty s : u^\alpha_I = s^\alpha{}_{,I}(x).)$$

Definition 8.37. Let $z = j^\infty_{x_0} s \in J^\infty E$, $x_0 \in M$. The *Cartan plane* C_z at z is the tangent space at z to the graph of $j^\infty s$, in other words $C_z = d(j^\infty s)(T_{x_0} M)$.

Exercise 8.38. Show that C_z is independent of s such that $z = j^\infty_x s$. In other words, the graphs of the ∞-jet prolongations of two sections s, t of E such that $s \sim^\infty_{x_0} t$ for some x_0 in their common domain, are tangent at the point $j^\infty_{x_0} s = j^\infty_{x_0} t$ (*Hint: show that if s is locally given by*

$$s : \{u^\alpha = s^\alpha(x)$$

(around x_0), then the tangent space at $z = j^\infty_{x_0} s$ to the graph of $j^\infty s$ is spanned by the tangent vectors

$$D_i|_z := \left.\frac{\partial}{\partial x^i}\right|_z + s^\alpha{}_{,Ii}(x_0) \left.\frac{\partial}{\partial u^\alpha_I}\right|_z = \left.\frac{\partial}{\partial x^i}\right|_z + u^\alpha_{Ii}(z) \left.\frac{\partial}{\partial u^\alpha_I}\right|_z,$$

so it only depends on z).

Definition 8.39. The *Cartan distribution* on $J^\infty E$ is the distribution C with $\Gamma(C)$ consisting of vector fields X on $J^\infty E$ such that, for all $z \in J^\infty E$, X_z belongs to the Cartan plane C_z.

Exercise 8.40. Show that C is a rank n, *involutive* distribution on $J^\infty E$ locally spanned by total derivatives

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_{Ii}\frac{\partial}{\partial u^\alpha_I}.$$

So the value at z of C is precisely the Cartan plane C_z. What is the shift of total derivatives? Finally, show that $\Gamma(C)$ consists of vector fields which are locally annihilated by one forms

$$du^\alpha_I - u^\alpha_{Ii}dx^i, \quad |I| \geq 0.$$

Remark 8.41. Notice that the Cartan distribution is π*-horizontal*, in the sense that $d\pi : T(J^\infty E) \to TM$ is injective, hence an isomorphism, when restricted to Cartan planes.

Exercise 8.42. Prove that

- a section σ of $J^\infty E$ is an *integral section* of C, i.e. $d\sigma(T_x M) = C_{\sigma(x)}$ for all $x \in M$, iff σ is the ∞-jet prolongation of a section E, and, more generally
- an n-dimensional submanifold S of $J^\infty E$ is an *integral submanifold* of C, i.e. $T_z S = C_z$ for all $z \in S$ iff, locally, S is the graph of the ∞-jet prolongation of a section of E.

Remark 8.43. Exercise 8.42 shows that *the Cartan distribution detects infinit jet prolongation*

8.3. **Infinitesimal Symmetries of the Cartan Distribution on $J^\infty E$.**

Remark 8.44. Vector fields on a profinite dimensional manifold need not to posses a flow. Consequently, the Frobenius Theorem may fail on a profinite dimensional manifold and an involutive distribution may possess several (locally maximal) or no integral submanifolds through a given point. As an instance notice that the graph of every ∞-jet prolongation is an integral submanifold of the Cartan distribution. As there are several local sections of E with the same ∞-jet at a given point, we conclude that there are several different (locally maximal) integral submanifolds through every point.

For the same reason, in the profinite dimensional setting, if one defines infinitesimal symmetries of a geometric structure via the Lie derivative, then infinitesimal symmetries cannot be integrated to finite ones in general, and the former can be "more many" than the latter. For this reason, on a profinite dimensional manifold equipped with a geometric structure, one does usually consider infinitesimal symmetries only.

Exercise 8.45. Show that total derivatives D_i do not possess a flow.

Definition 8.46. An *infinitesimal symmetry* of the Cartan distribution C on $J^\infty E$ is a vector field $X \in \mathfrak{X}(J^\infty E)$ such that $[X, \Gamma(C)] \subseteq \Gamma(C)$. A *non-trivial symmetry* is an infinitesimal symmetry modulo $\Gamma(C)$. An *evolutionary vector field* is a *vertical* infinitesimal symmetry, i.e. an infinitesimal symmetry X such that $X \circ \pi^* = 0$, or, equivalently, $d\pi(X_z) = 0$ for all $z \in J^\infty E$.

Remark 8.47. Infinitesimal symmetries form a Lie algebra, denoted $\mathfrak{X}_C$, under the commutator. $\Gamma(C)$ is an ideal in $\mathfrak{X}_C$. Accordingly, non-trivial symmetries form a Lie algebra $\mathfrak{X}_C/\Gamma(C)$ sometimes denoted by $\operatorname{sym} E$.

Evolutionary vector fields do also form a Lie algebra, actually a Lie subalgebra of $\mathfrak{X}_C$, denoted $V\mathfrak{X}_C$.

Lemma 8.48. *The inclusion $V\mathfrak{X}_C \hookrightarrow \mathfrak{X}_C$ induces a Lie algebra isomorphism $i : V\mathfrak{X}_C \simeq \operatorname{sym} E$.*

Proof. Let $X \in V\mathfrak{X}_C$, and let $\chi = X \operatorname{mod} \Gamma(C)$ be the corresponding non-trivial symmetry. If $\chi = 0$, then $X \in \Gamma(C)$. As C is π-horizontal this in turn implies that $X = 0$. This shows that i is injective. To prove surjectivity, let $\chi = X \operatorname{mod} \Gamma(C)$ be a non-trivial symmetry, $X \in \mathfrak{X}_C$. As C is π-horizontal, X can be uniquely written in the form $X = CX + VX$ with $CX \in \Gamma(C)$ and VX a *vertical* vector field, i.e. $X \circ \pi^* = 0$. VX is an infinitesimal symmetry, indeed

$$[VX, \Gamma(C)] = [X - CX, \Gamma(C)] \subseteq [X, \Gamma(C)] + [CX, \Gamma(C)] \subseteq \Gamma(C).$$

Clearly $i(VX) = \chi$. □

In the following we will consider vector fields relative to the projection $p_0 : J^\infty E \to E$, i.e. derivations $\chi : C^\infty(E) \to C^\infty(J^\infty E)$. We will focus on those derivations $\chi : C^\infty(E) \to C^\infty(J^\infty E)$ which are *vertical*, in the sense that $\chi \circ \pi^* = 0$. The latter will be referred to as *generating sections of higher symmetries* for reasons that will be more clear after the following discussion. Generating sections form a $C^\infty(J^\infty E)$-module, denoted $\varkappa$, in an obvious way. $\varkappa$ can be interpreted geometrically as the module of sections of the induced vector bundle $p_0^* VE$.

Exercise 8.49. Define rigorously the profinite dimensional manifold $p_0^* VE$ (together with the projection $p_0^* VE \to J^\infty E$), and show that generating sections are equivalent to sections of $p_0^* VE \to J^\infty E$.

Theorem 8.50. *There is a canonical $\mathbb{R}$-linear one-to-one correspondence between*

(1) *evolutionary vector fields,*
(2) *generating sections of higher symmetries.*

Proof. Let $\mathcal{\partial}$ be an evolutionary vector field. Put

$$\chi := \mathcal{\partial}|_{C^\infty(E)} : C^\infty(E) \to C^\infty(J^\infty E).$$

It follows from the fact that $\mathcal{\partial}$ is vertical, that χ is vertical as well. Actually, χ determines $\mathcal{\partial}$ completely. To see this, work locally. $\mathcal{\partial}$ is locally given by

$$\mathcal{\partial} = \chi^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{|I|>0} \mathcal{\partial}_I^\alpha \frac{\partial}{\partial u_I^\alpha}.$$

Then χ is locally given by

$$\chi = \chi^\alpha \frac{\partial}{\partial u^\alpha}.$$

If l is the shift of $\mathcal{Э}$, then $\mathcal{Э}^\alpha_I$ depend on derivatives up to order $|I|+l$. In particular, χ^α depend on derivatives up to order l.

Now, notice that $[\mathcal{Э}, D_i] = 0$ for all $i = 1, \ldots, n$. Indeed, from $D_i \in \Gamma(C)$ and $\mathcal{Э} \in \mathfrak{X}_C$, it follows $[\mathcal{Э}, D_i] \in \Gamma(C)$. On another hand

$$[\mathcal{Э}, D_i] \circ \pi^* = \mathcal{Э} \circ D_i \circ \pi^* - D_i \circ \mathcal{Э} \circ \pi^* = \mathcal{Э} \circ \pi^* \circ \frac{\partial}{\partial x^i} = 0.$$

So, $[\mathcal{Э}, D_i]$ is vertical. As C is π-horizontal, it follows that $[\mathcal{Э}, D_i] = 0$. Now compute

$$\mathcal{Э}^\alpha_I = \mathcal{Э}(u^\alpha_I) = \mathcal{Э}(D_I u^\alpha) = D_I \mathcal{Э}(u^\alpha) = D_I \chi^\alpha,$$

where, for $I = i_1 \cdots i_k$, we put $D_I := D_{i_1} \circ \cdots \circ D_{i_k}$. This shows that, locally,

$$\mathcal{Э} = \sum_{|I| \geq 0} D_I \chi^\alpha \frac{\partial}{\partial u^\alpha_I} \tag{21}$$

which is completely determined by χ as claimed.

Now, given a generating section χ, it is easy to see that Formula (21) defines a local evolutionary vector field, and all these local evolutionary vector fields glue together to a well-defined global evolutionary vector field (Exercise 8.51). This concludes the proof. □

Exercise 8.51. Show that Formula (21) defines a local evolutionary vector field, and all these local evolutionary vector fields glue together to a well-defined global evolutionary vector field.

The evolutionary vector field corresponding to a generating section $\chi \in \varkappa$ is denoted by $\mathcal{Э}_\chi$. Formula (21) shows that if χ is locally given by

$$\chi = \chi^\alpha \frac{\partial}{\partial u^\alpha}, \quad \chi^\alpha = \chi^\alpha(x, \ldots, u_I, \ldots)$$

then $\mathcal{Э}_\chi$ is locally given by

$$\mathcal{Э}_\chi = \sum_{|I| \geq 0} D_I \chi^\alpha \frac{\partial}{\partial u^\alpha_I}.$$

In particular, the shift of $\mathcal{Э}_\chi$ is the maximum order of a derivative appearing in $\chi^\alpha(x, \ldots, u_I, \ldots)$.

Corollary 8.52. *There is a vector space isomorphism* $\operatorname{sym} E = \varkappa$.

It follows from the above corollary that $\operatorname{sym} E$ is a $C^\infty(J^\infty E)$-module in a canonical way. More importantly, $\varkappa$ is a Lie algebra in a canonical way. The Lie bracket on $\varkappa$ is denoted by

$$\{-,-\} : \varkappa \times \varkappa \to \varkappa$$

and called the *higher Jacobi bracket*. It is implicitly given by

$$[\mathcal{Э}_\chi, \mathcal{Э}_\psi] = \mathcal{Э}_{\{\chi,\psi\}}$$

for all $\chi, \psi \in \varkappa$.

Exercise 8.53. Prove that the higher Jacobi bracket is locally given by

$$\{\chi, \psi\} = \left(\mathcal{\ni}_\chi \psi^\alpha - \mathcal{\ni}_\psi \psi^\alpha\right) \frac{\partial}{\partial u^\alpha} = \sum_{|I| \geq 0} \left(D_I \chi^\beta \frac{\partial \psi^\alpha}{\partial u^\beta_I} - D_I \psi^\beta \frac{\partial \chi^\alpha}{\partial u^\beta_I} \right) \frac{\partial}{\partial u^\alpha}.$$

DipMat, Università degli Studi di Salerno & Istituto Nazionale di Fisica Nucleare, GC Salerno, via Giovanni Paolo II n° 123, 84084 Fisciano (SA) Italy.

Email address: `lvitagliano@unisa.it`