Characteristics, Bicharacteristics, and Geometric Singularities of Solutions of PDEs

Lecture II: Singularities of Solutions of PDEs

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In the first lecture, I reviewed the classical theory of characteristic Cauchy data for PDEs. The concept of a PDE itself has a nice intrinsic, geometric formulation which is manifestly coordinate independent. The key concept in such geometric formulation is that of jet spaces. Basically, a jet space is a manifold whose points are Taylor polynomials of functions. Characteristics can be defined intrinsically within jet spaces. As usual, also in this context, the geometric approach clarifies, unifies, and allows for interesting generalizations.

Plan of the Second Lecture

I will review the fundamentals of the jet space approach to PDEs. I will also give geometric definitions of

- singularities of solutions,
- characteristics of PDEs,

and discuss their relationship, already sketched in the first lecture.
Let $\pi : E \rightarrow M$ be a fiber bundle, and $(x,u)$ a bundle chart on it. A (local) section $\sigma$ of $\pi$ is locally given by

$$\sigma : u = f(x).$$

Jet spaces formalize geometrically the concept of Taylor polynomial of a section of $\pi$.

$$\frac{\partial^{|I|} f}{\partial x^I}(x_0) = \frac{\partial^{|I|} f'}{\partial x^I}(x_0), \quad |I| \leq k.$$ 

The $k$-th jet space is the manifold $J^k$ of classes of tangency up to the order $k$. It is coordinatized by $(x, u_I, \ldots), |I| \leq k$. A section $\sigma$ of $\pi$ can be prolonged to a section $j^k \sigma$ of $J^k \rightarrow M$. $j^k \sigma$ is locally given by

$$j^k \sigma : u_I = \frac{\partial^{|I|} f}{\partial x^I}(x), \quad |I| \leq k,$$

and encodes "derivatives of $\sigma$ up to the order $k". \\\nLuca Vitagliano \hspace{7cm} Singularities of Solutions of PDEs
The Cartan Distribution

There is a tower of fiber bundles

\[ M \xleftarrow{\pi} E \xleftarrow{\pi_1} J^1 \xleftarrow{\pi_k} \cdots \xleftarrow{\pi_k} J^k \xleftarrow{\pi_{k+1}} J^{k+1} \xleftarrow{\pi_k} \cdots \]

Any tangent space to the image of a holonomic section \( j^k \sigma \) is an \( R \)-plane.

Definition: Cartan Distribution

\[ C : J^k \ni \theta \longmapsto C_\theta := \langle R \text{-planes through } \theta \rangle \subset T_\theta J^k. \]

In local coordinates

\[ C = \langle \ldots, D_i, \ldots, \partial/\partial u_I, \ldots \rangle_{|I|=k}, \]

where \( D_i := \frac{\partial}{\partial x^i} + \sum_{|I|<k} u_{Ii} \frac{\partial}{\partial u_I} \) are total derivatives. Dually

\[ \text{Ann } C = \langle \ldots, du_I - u_{Ii} dx^i, \ldots \rangle_{|I|<k}. \]

Holonomic sections are integral sections of the Cartan distribution.
Jet spaces allow a coordinate-free definition of PDEs.

**Definition: System of PDEs**

A system of $r$ (nonlinear) PDEs of order $k$ imposed on sections of the bundle $\pi$ is a codimension $r$ submanifold $\mathcal{E} \subset J^k$. A solution of $\mathcal{E}$ is a section $\sigma$ of $\pi$ such that $\text{im} \ j^k \sigma \subset \mathcal{E}$.

In local coordinates

$$\mathcal{E} : F(x, \ldots, u_I, \ldots) = 0, \quad |I| \leq k,$$

and a section $\sigma : u = f(x)$ is a solution iff $F(x, \ldots, \partial^{|I|} f / \partial x^I, \ldots) = 0$.

**Definition: Cartan Distribution of a PDE**

$$C(\mathcal{E}) := C \cap T\mathcal{E}.$$  

Solutions of $\mathcal{E}$ identify (via prolongations) with integral sections of $C(\mathcal{E})$. 
Jets of Submanifolds

There are many cases when a PDE is imposed on a submanifold. *Jets of submanifolds can be defined similarly to jets of sections of bundles.*

Let $E$ be an $(n + m)$-manifold. A submanifold $L \subset E$ is locally given by

$$L : u = f(x), \quad (x, u) \text{ a divided chart.}$$

Two $n$-submanifolds $L, L'$ are tangent at $e_0 \in L \cap L'$ up to the order $k$ if

$$\frac{\partial^{|I|} f}{\partial x^I}(e_0) = \frac{\partial^{|I|} f'}{\partial x^I}(e_0), \quad |I| \leq k.$$

The $k$-th jet space (of $n$-submanifolds of $E$) is the manifold $J^k(E, n)$ of classes of tangency up to the order $k$.

**Remark: First Order PDEs Imposed on Submanifolds**

$J^1(E, n) = \text{Gr}(TE, n)$. Accordingly, a *first order PDE imposed on $n$-submanifolds of $E$* is a submanifold $\mathcal{E} \subset \text{Gr}(TE, n)$, and a *solution* is an $n$-submanifold $L \subset E$ whose tangent spaces belong to $\mathcal{E}$.
Some kind of singularities of solutions can be treated using jets.

**Remark: Horizontal Integral Manifolds of $C$**

Sections of $\pi$ (locally) identify with (locally) maximal integral $n$-manifolds of $C$ that are *horizontal* with respect to $\pi_k : J^k \rightarrow J^{k-1}$, but generic maximal integral $n$-manifolds need not to be horizontal.

**Definition: Singular Sections**

A *singular section* of $\pi$ is a maximal integral $n$-manifold $N$ of $C$.

*Singular solutions of a PDE $\mathcal{E}$* are singular sections contained into $\mathcal{E}$.

*Singular points* are singularities of the map $\pi_k|_N : N \rightarrow J^{k-1}$:

$$\text{sing } N := \{ \theta \in N : \text{rank } d_\theta \pi_k|_N < n \}$$

and can be classified along the Thom-Boardmann theory.
Examples of Singular Solutions

Example
Let $\mathcal{E} : u_x^2 - \frac{9}{4}x = 0$. 
The curve
\[
\begin{align*}
    \{ & u^2 - x^3 = 0 \\
        & u_x^2 - \frac{9}{4}x = 0 
\end{align*}
\]
is a singular solution.

Example
Let $\mathcal{E} : u_x^2 + x^2 - 1 = 0$. 
The curve
\[
\begin{align*}
    \{ & x = \sin 2t \\
        & u = \frac{1}{2}t + \frac{1}{8}\sin 4t \\
        & u_x = \cos 2t 
\end{align*}
\]
is a singular solution.
Singular $R$-planes are the \textit{infinitesimal version} of singular sections.

**Definition: Singular $R$-planes**

A \textit{singular} $R$-plane is the tangent space to a singular section at a singular point. The \textit{type} of a singular $R$-plane $K$ at $\theta$ (also called, the \textit{type of the singular point $\theta$}) is \( \text{type } K := \dim \ker d\pi_k|_K \).

**Remark: Metaplectic Structure of $C$**

(Singular) $R$-planes have a useful characterization in terms of the \textit{metaplectic structure} of $C$:

\[ \Omega : C \times C \longrightarrow TJ^k/C, \quad (X, Y) \longmapsto [X, Y] + C. \]

In local coordinates

\[ \Omega = \sum_{|I|<k} du_{Ii} \wedge dx^i \otimes \frac{\partial}{\partial u_I}. \]

(Singular) $R$-planes are $n$-dimensional, isotropic subspaces of $\Omega$, i.e., subspaces $K \subset C$ such that $\Omega(\xi, \eta) = 0$ for all $\xi, \eta \in K$. 
Let $N$ be a singular section. The *singular locus* $\text{sing } N$ is *stratified* by singular points of different type. For simplicity, assume
- $\text{sing } N$ is a smooth submanifold,
- type $\theta$ is constant along $\text{sing } N$,
- $\ker d\pi_k|_{T_\theta N}$ is transversal to $T_\theta \text{sing } N$.

It follows that the *type of singular points is $\text{codim sing } N$.*

**Definition: Fold Type Singularities**

A singular section $N$ has a *fold-type singularity* if $\text{codim sing } N = 1$. 

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The shape of (fold-type) singularities is governed by a suitable PDE. Let $\mathcal{E} \subset j^k$ be a PDE.

**Definition: Fold-Type Singularity Equation**

The *fold-type singularity equation associated to* $\mathcal{E}$ *is*

$$\Sigma_1 \mathcal{E} := \{(d\pi_k)(K) : K \text{ is a type 1 singular } R\text{-plane tangent to } \mathcal{E}\}.$$
Since $\dim K = n - 1$ for all $K \in \Sigma_1 \mathcal{E}$, then
\[
\Sigma_1 \mathcal{E} \subset \text{Gr}(TJ^{k-1}, n - 1) = J^1(J^{k-1}, n - 1)
\]
is a first order PDE for $(n - 1)$-submanifolds of $J^{k-1}$.

**Remark**

If $N$ is a singular solution of $\mathcal{E}$ with a fold-type singularity, then $\pi_k(\text{sing } L)$ is a solution of $\Sigma_1 \mathcal{E}$. Thus, $\Sigma_1 \mathcal{E}$ describes the shape of fold-type singularity loci of solutions of $\mathcal{E}$.

**Remark**

Solutions of $\Sigma_1 \mathcal{E}$ are automatically integral submanifolds of the Cartan distribution on $J^{k-1}$. In their turn, integral $(n - 1)$-submanifolds of $J^{k-1}$ are naturally interpreted as Cauchy data for $\mathcal{E}$. Thus, $\Sigma_1 \mathcal{E}$ is a PDE imposed on Cauchy data for $\mathcal{E}$. 
Symbol of a Non-Linear PDE

Characteristic covectors of a PDE can be defined in terms of jets.

**Proposition**

The bundle $\pi_k : J^k \to J^{k-1}$ is an affine bundle with model vector bundle

$$S^k T^* M \otimes_{J^{k-1}} VE \to J^{k-1}.$$ 

Quasi-linear $k$-th order PDEs are affine subbundles of $\pi_k : J^k \to J^{k-1}$.

**Definition: Symbol of a Non-Linear PDE**

The *symbol* of a PDE $\mathcal{E} \subset J^k$ at a point $\theta \in \mathcal{E}$ is

$$g_\theta := T_\theta \mathcal{E} \cap \ker d_\theta \pi_k \subset S^k T^*_x M \otimes V_{\mathcal{E}}.$$ 

If, locally, $\mathcal{E} : F(x, \ldots, u_I, \ldots) = 0$, and $v = v_{i_1 \ldots i_k} dx^{i_1} \cdots dx^{i_k} \otimes \frac{\partial}{\partial u} \in S^k T^*_x M \otimes V_{\mathcal{E}}$, then $v \in g_\theta$ iff

$$\left. \frac{1}{I!} \frac{\partial F}{\partial u_{i_1 \ldots i_k}} \right|_\theta \cdot v_{i_1 \ldots i_k} = 0.$$
Characteristic Covectors of a Non-Linear PDE

**Definition: Characteristic Covectors**

A non-zero covector \( p \in T^* \mathcal{M} \) is characteristic for \( \mathcal{E} \) at \( \theta \in \mathcal{E} \) if there exists \( \zeta \in V_{e \mathcal{E}} \) such that

\[
p^k \otimes \zeta \in g_{\theta}.
\]

In local coordinates, \( p = p_i dx^i \) is characteristic iff

\[
\text{rank} \left( \begin{array}{c}
\frac{1}{I!} p_{i_1} \cdots p_{i_k} \\
\frac{\partial F}{\partial u_{i_1 \cdots i_k}} \bigg|_{\theta}
\end{array} \right) < m. \quad (CC)
\]

For a determined \((r = m)\), quasi-linear system \( \mathcal{E} \) locally given by

\[
\mathcal{E} : A^{i_1 \cdots i_k} u_{i_1 \cdots i_k} = g,
\]

(CC) reduces to

\[
\det A(p) = 0, \quad A(p) := p_{i_1} \cdots p_{i_k} A^{i_1 \cdots i_k}.
\]

**Remark**

The classical definition is recovered.
Recall that *singularities of solutions occur along characteristic surfaces*. This *informal statement* can be made rigourous and coordinate-free within the jet space approach to PDEs.

- $E \subset J^k$ a PDE, and $\theta = (j^k \sigma)(x) \in E$,
- $p \in T^*_x M$, $p \neq 0$,
- $\ker \theta p := (d_x j^{k-1} \sigma)(\ker p) \in \text{Gr}(TJ^{k-1}, n - 1)$.

**Theorem**

*If $E$ is formally integrable, then the equation of fold-type singularities is “dual” to the manifold of characteristic covectors, i.e.,*

$$\Sigma_1 E = \{ \ker \theta p : p \text{ is characteristic for } E \text{ at } \theta \}.$$

*Moreover, when interpreted as an equation for Cauchy data, $\Sigma_1 E$ is the equation for compatible, characteristic Cauchy data.*
An Example: The 2D Klein-Gordon Equation

The equation for characteristic surfaces does not contain a full information on the original equation: On the other hand, the equation for singularities may contain a full information on the original equation.

The 2D flat Klein-Gordon equation is

$$\mathcal{E}_{KG} : u_{tt} - u_{xx} + m^2 u = 0.$$ 

Coordinatize $J^1(J^1, 1)$ by $x, t, u, u_x, u_t, t', u', u'_x, u'_t$, with $f' := df/dx$. Then

$$\Sigma_1 \mathcal{E}_{KG} : \left\{ \begin{array}{l}
(t')^2 = 1 \\
u' = u_x + t'u_t \\
u'_x = m^2 u + t'u'_t
\end{array} \right. . $$

Eliminating $u_x$ one gets a system for the Cauchy data $t, u, u_t$:

$$\left\{ \begin{array}{l}
(t')^2 = 1 \\
u'' - t''u_t - 2t'u'_t = m^2 u
\end{array} \right. . $$

$\Sigma_1 \mathcal{E}_{KG}$ contains the mass parameter $m$. 
Let $\mathcal{E} \subset J^k$ be a determined system of PDEs, and $N$ a singular solution with a fold-type singularity along $\text{sing } N$. Interpret $N$ as a wave propagating in the space-time $M$. Then the projection $\text{sing } N$ of $\text{sing } N$ on $M$ is the wave-front, and it is a characteristic surface.

**Theorem**

sing $N$ is equipped with a canonical field of directions.

**Corollary**

The wave-front $\text{sing } N$ is foliated by 1-dimensional manifolds.

**Definition: Bicharacteristic Lines on Wave Fronts**

The 1-dimensional leaves of $\text{sing } N$ are bicharacteristics.

Wave-fronts propagate along bicharacteristics!
The theory of PDEs can be formulated in a manifestly coordinate-free way on jet spaces. One can give very general, geometric definitions of singularities of solutions, and characteristics. Fold-type singularities are dual to characteristics. Their shape is governed by a fold-type singularity equation which describes compatible, characteristic Cauchy data.

The wave-front of a singular solution is a characteristic surface foliated by bicharacteristic lines, i.e., wave-fronts propagate along bicharacteristics. I will not discuss bicharacteristics in full generality. I will only consider the special case when the symbol does only depend on points of the base manifold $M$ (e.g., linear equations). In this case, characteristic surfaces are solutions of a first order, scalar PDE, which can be treated with the method of characteristics. Bicharacteristics are then characteristics of the characteristic equation.

From a physical point of view, the passage from the wave-front to its bicharacteristics can be interpreted as the passage from wave optics (the dynamics of wave-fronts) to geometric optics (the dynamics of rays), or from wave (quantum) mechanics to classical mechanics.
Generalities on Jet Spaces


References

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