Lie groupoids can be seen as *smooth atlases on stacks over smooth manifolds* [BEHREND, XU 2008]. In other words a Lie groupoid $G \Rightarrow M$ equips its orbit space $M/G$ with a structure of “generalized manifold” (e.g. orbifolds, leaf spaces, orbit spaces, etc.).

**Remark**

*Multiplicative vector fields* on a Lie groupoid $G \Rightarrow M$ serve as representatives for *vector fields on the differentiable stack* $[M/G]$ [ORTIZ, WALDRON 2017].

VB groupoids are *vector bundle objects in the category of Lie groupoids* and can be seen as *linear atlases on vector bundles over differentiable stacks* [DEL HOYO, ORTIZ 2016].

**Aim**

The aim of the talk is discussing *multiplicative derivations of VB groupoids*, and their infinitesimal counterparts (*VB algebroids*). Hopefully they will serve as representatives for *derivations of vector bundles over differentiable stacks*. 
1. Multiplicative Vector Fields
2. VB Groupoids and Algebroids
3. Multiplicative Derivations
4. Morphic Derivations
5. Deformation Complexes
1. Multiplicative Vector Fields
2. VB Groupoids and Algebroids
3. Multiplicative Derivations
4. Morphic Derivations
5. Deformation Complexes
Let $G \rightarrow M$ be a Lie groupoid with Lie algebroid $A \rightarrow M$.

**Remark**

$TG \rightarrow TM$ is a Lie groupoid as well.

**Proposition [Mackenzie, Xu 1998]**

Let $X \in \mathfrak{X}(G)$. The following two conditions are equivalent:

1. $X$ generates a flow of groupoid automorphisms,
2. $X : G \rightarrow TG$ is a groupoid map.

**Definition**

A vector field $X \in \mathfrak{X}(G)$ is *multiplicative* if either 1 or 2.

**Example/Definition**

For $\alpha \in \Gamma(A)$, $\overleftarrow{\alpha} + \overrightarrow{\alpha} \in \mathfrak{X}(G)$ is an *internal* multiplicative vector field.
Let $E \to M$ be a vector bundle (VB). $TE \to TM$ is a VB as well.

**Proposition**

Let $X \in \mathfrak{X}(E)$. The following three conditions are equivalent:

1. $X$ generates a flow of VB automorphisms,
2. $X : E \to TE$ is a VB map (over some map $M \to TM$),
3. The Lie derivative along $X$ preserves constant vector fields.

**Definition**

A vector field $X \in \mathfrak{X}(E)$ is linear if either 1, 2, or 3.

**Remark**

There are various further characterizations of linear vector fields via coordinates, or linear functions on $E$, or the Euler vector field of $E$, ...
Linear vector fields

Definition

A derivation of $E$ is an $\mathbb{R}$-linear operator $\delta : \Gamma(E) \to \Gamma(E)$ such that

$$\delta(f \epsilon) = \sigma(\delta)(f) \epsilon + f \delta \epsilon$$

for some $\sigma(\delta) \in \mathfrak{X}(M)$.

$e \in \Gamma(E), f \in C^\infty(M)$. Vector field $\sigma(\delta)$ is the symbol of $\delta$.

Let $p : E \to M$ be a VB and $\epsilon \in \Gamma(E)$. The constant vector field on $E$ corresponding to $\epsilon$ is $\epsilon^V$:

$$\epsilon^V_e = \frac{d}{dt}|_{t=0}(e + t\epsilon_{p(e)}), \quad e \in E$$

Proposition

There is a 1-to-1 correspondence between linear vector fields on $E$, and derivations of $E$, given by

$$X \mapsto \delta_X, \quad (\delta_X \epsilon)^V = [X, \epsilon^V].$$
Let $A \Rightarrow M$ be a Lie algebroid. $TA \Rightarrow TM$ is a Lie algebroid as well.

**Reminder**

A *Lie algebroid derivation* of $A$ is a derivation $\delta$ s.t.

$$\delta[\alpha, \beta]_A = [\delta\alpha, \beta]_A + [\alpha, \delta\beta]_A, \quad \alpha, \beta \in \Gamma(A),$$

e.g. an *internal derivation* $[\alpha, -]_A$.

**Proposition [Mackenzie, Xu 1998]**

Let $X \in \mathfrak{X}(A)$. The following three conditions are equivalent:

1. $X$ generates a flow of Lie algebroid automorphisms,
2. $X : A \rightarrow TA$ is an algebroid map (over some map $M \rightarrow TM$),
3. $X$ is linear and $\delta_X$ is a Lie algebroid derivation.

**Definition**

A vector field $X \in \mathfrak{X}(A)$ is IM if either 1, 2, or 3.
Let $G$ be a Lie groupoid and let $A$ be its Lie algebroid.

**Theorem [Mackenzie, Xu 1998]**

The Lie functor maps multiplicative vector fields to IM vector fields.

If $G$ is s-simply-connected we get a 1-to-1 correspondence:

\[
\{\text{Multiplicative vector fields on } G\} \cong \{\text{IM vector fields on } A\} \cong \{\text{Lie algebroid derivations of } A\}
\]

Lie algebroid derivation $\delta$ corresponding to multiplicative vector field $X$ is:

\[
\overrightarrow{\delta \alpha} = [X, \overrightarrow{\alpha}], \quad \alpha \in \Gamma(A).
\]

Can we get the analogue for VB groupoids?
1. Multiplicative Vector Fields
2. VB Groupoids and Algebroids
3. Multiplicative Derivations
4. Morphic Derivations
5. Deformation Complexes
VB groupoids are VB objects in the category of Lie groupoids.

**Definition [Pradines 1988]**

A VB groupoid $\Omega \rightarrow E$ over a Lie groupoid $G \rightarrow M$ is a diagram:

\[
\begin{array}{ccc}
\Omega & \rightarrow & E \\
\downarrow & & \downarrow \\
G & \rightarrow & M
\end{array}
\]

also written $(\Omega, E \downarrow G, M)$ such that

- the rows are Lie groupoids,
- the columns are VBs,
- all VB structure maps are Lie groupoid maps.

VB groupoid maps are defined in the obvious way.

**Example**

$T G \rightarrow T M$ is a VB groupoid over $G \rightarrow M$. 
Definition [Mackenzie 1992]

A double VB is a diagram:

\[
\begin{array}{ccc}
W & \rightarrow & E \\
\downarrow & & \downarrow \\
A & \rightarrow & M
\end{array}
\]

also written \((W, E \| A, M)\), such that

- rows and columns are VBs,
- the vertical VB structure maps are morphisms of (horizontal) VBs.

Double VB maps are defined in the obvious way.

Example

If \(A \rightarrow M\) is a VB, then \((TA, TM \| A, M)\) is a double VB.
Consider a double VB:

\[
\begin{array}{ccc}
W & \rightarrow & E \\
\downarrow & & \downarrow q \\
A & \rightarrow & M
\end{array}
\]

**Definition**

The *core* of \((W, E \parallel A, M)\) is the following VB over \(M\):

\[
C := \ker(W \rightarrow E) \cap \ker(W \rightarrow A).
\]

**Remark**

A section \(\chi \in \Gamma(C)\) determines a section \(\hat{\chi}\) of \(W \rightarrow E\) via:

\[
\hat{\chi}_e := (0_{W,E})_e +_{W,A} \chi_q(e), \quad e \in E.
\]
Consider a double VB:

\[
\begin{array}{c}
W 
\downarrow \\
A 
\end{array}
\quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\downarrow \\
\downarrow \\
E 
\downarrow \\
M 
\end{array}
\]

**Definition**

- **Linear sections** of \( W \rightarrow E \) are those which are also VB maps.
- **Core sections** of \( W \rightarrow E \) are those of the form \( \hat{\chi} \), with \( \chi \in \Gamma(C) \).

**Proposition [Fat VB]**

**Linear sections** of \( W \rightarrow E \) are sections of a VB \( \hat{A} \rightarrow M \) sitting in a short exact sequence of VBs:

\[
0 \rightarrow \text{Hom}(E, C) \rightarrow \hat{A} \rightarrow A \rightarrow 0
\]
VB algebroids are VB objects in the category of Lie algebroids.

**Definition**

A VB algebroid $W \rightarrow E$ over a Lie algebroid $A \rightarrow M$ is a double VB:

$$
\begin{array}{ccc}
W & \rightarrow & E \\
\downarrow & & \downarrow \\
A & \rightarrow & M
\end{array}
$$

such that

- the rows are Lie algebroids,
- all (vertical) VB structure maps are Lie algebroid maps.

VB algebroid maps are defined in the obvious way.

**Example**

If $A \rightarrow M$ is a Lie algebroid, then $(TA, TM \downarrow A, M)$ is a VB algebroid.
Proposition

In a VB algebroid $W \to E$ over a Lie algebroid $A \to M$:

- $\text{[linear,linear]}_W \subset \text{linear}$,
- $\text{[linear,core]}_W \subset \text{core}$,
- $\text{[core,core]}_W = 0$.

- $\rho_W(\text{linear}) \subset \text{linear}$,
- $\rho_W(\text{core}) \subset \text{constant}$,

Corollary [Algebraic data of $(W, E \parallel A, M)$]

- $0 \to \text{Hom}(E, C) \to \hat{A} \to A \to 0$ is a SES of Lie algebroids,
- $E, C$ are equipped with flat $\hat{A}$-connections $\nabla^E, \nabla^C$,
- there is a VB map $\partial : C \to E$ (core-anchor).

Additionally, for all $\hat{\alpha} \in \Gamma(\hat{A})$, and all $h \in \Gamma(\text{Hom}(E, C))$

- $\text{[}\hat{\alpha}, h\text{]}_{\hat{A}} = \nabla^C_{\hat{\alpha}} \circ h - h \circ \nabla^E_{\hat{\alpha}}$,
- $\nabla^E_h = \partial \circ h$,
- $\nabla^C_h = -h \circ \partial$. 
Differentiation and Integration

Consider a VB groupoid

\[ \Omega \xrightarrow{\tilde{s}} E \xleftarrow{\tilde{t}} \]
\[ \downarrow \quad \downarrow \]
\[ G \xrightarrow{} M \]

Definition

The core of \((\Omega, E \parallel G, M)\) is the following VB over \(M\)

\[ C := \ker \tilde{s}|_M \]

Proposition [Bursztyn, Cabrera, del Hoyo 2016]

- The Lie functor maps VB groupoids to VB algebroids with the same core, and VB groupoid maps to VB algebroid maps.
- Let \((W, E \parallel A, M)\) be a VB algebroid and let \(\Omega\) be an \(s\)-simply-connected integration of \(W\). Then \(A\) also integrates to some \(s\)-simply-connected \(G\), and \((\Omega, E \parallel G, M)\) is a VB groupoid.
Outline

1. Multiplicative Vector Fields
2. VB Groupoids and Algebroids
3. Multiplicative Derivations
4. Morphic Derivations
5. Deformation Complexes
Consider a VB groupoid and its VB algebroid

\[
\begin{align*}
\Omega & \xrightarrow{\delta} E \\
\downarrow & \quad \downarrow \\
\mathcal{G} & \xrightarrow{\delta} M
\end{align*}
\quad \text{and} \quad
\begin{align*}
W & \xrightarrow{\delta} E \\
\downarrow & \quad \downarrow \\
A & \xrightarrow{\delta} M
\end{align*}
\]

**Definition**

An *infinitesimal automorphism* \( X \) of \((\Omega, E \Join \mathcal{G}, M)\) is both
- a *multiplicative* vector field for \( \Omega \Rightarrow E \),
- a *linear* vector field for \( \Omega \rightarrow \mathcal{G} \).

In particular, it defines a derivation \( \delta_X \) of \( \Omega \rightarrow \mathcal{G} \) called *multiplicative*.

**Example/Definition**

For \( \hat{\alpha} \in \Gamma(\hat{A}) \), \( \hat{\alpha} + \overleftarrow{\alpha} \) is an *internal* infinitesimal automorphism.
Consider a double VB:

\[
\begin{array}{ccc}
W & \rightarrow & E \\
\downarrow & & \downarrow \\
A & \rightarrow & M
\end{array}
\]

**Definition**

An *infinitesimal automorphism* $X$ of $(W, E \parallel A, M)$ is a vector field $X \in \mathfrak{X}(W)$ linear wrt to both $W \rightarrow E$, and $W \rightarrow A$. In particular it defines

- a derivation $\delta_{X, \text{hor}}$ of $W \rightarrow E$ (*horizontal derivation*),
- a derivation $\delta_{X, \text{ver}}$ of $W \rightarrow A$ (*vertical derivation*).

**Example**

The Euler vector fields of both $W \rightarrow E$ and $W \rightarrow A$ are infinitesimal automorphisms.
Proposition

Let $X$ be an infinitesimal automorphism of double VB $(W, E \| A, M)$. Then

- $\delta_{X, \text{hor}}(\text{linear}) \subset \text{linear}$,
- $\delta_{X, \text{hor}}(\text{core}) \subset \text{core}$,
- $[\sigma(\delta_{X, \text{hor}}), \text{constant}] \subset \text{constant}$.

Corollary [Algebraic data of $X$]

- $\delta_{X, \text{hor}}$ induces derivations $\delta_{\hat{A}}, \delta_{C}, \delta_{E}$ of $\hat{A}, C, E$, sharing the same symbol. Additionally, recall

$$0 \to \text{Hom}(E, C) \to \hat{A} \to A \to 0.$$  

Then, for all $\hat{\alpha} \in \Gamma(\hat{A})$, and all $h \in \Gamma(\text{Hom}(E, C))$

- $\delta_{\hat{A}}h = \delta_{C} \circ h - h \circ \delta_{E}$.

Correspondence $X \mapsto (\delta_{\hat{A}}, \delta_{C}, \delta_{E})$ is 1-to-1.
Consider a VB algebroid

\[ W \xrightarrow{\longrightarrow} E \]

\[ \downarrow \quad \quad \downarrow \]

\[ A \xrightarrow{\longrightarrow} M \]

**Definition**

An *infinitesimal automorphism* \( X \) of \((W, E \downarrow A, M)\) is both

- an IM vector field for \( W \Rightarrow E \),
- a linear vector field for \( W \to A \).

In particular, it defines a derivation \( \delta_{X, \text{ver}} \) of \( W \to A \) called IM.

**Example/Definition**

For \( \hat{\alpha} \in \Gamma(\hat{A}) \), *internal* derivation \( [\hat{\alpha}, -]_W \) corresponds to an *internal* infinitesimal automorphism, hence to an IM derivation.
Algebraic Data of Infinitesimal Automorphisms

Theorem

Let $X$ be an infinitesimal automorphism of VB algebroid $(\mathcal{W}, E \parallel A, M)$. In particular it is an infinitesimal automorphism of a double VB. So it corresponds to $(\delta_{\hat{A}}, \delta_C, \delta_E)$. Recall that, in this case,

- $\hat{A}$ is a Lie algebroid acting on $E, C$ via flat connections $\nabla^E, \nabla^C$,
- There is a core-anchor $\partial : C \to E$.

Then

- $\delta_{\hat{A}}$ is a Lie algebroid derivation,
- $[\delta_E, \nabla^E_{\hat{a}}] = \nabla^E_{\delta_{\hat{A}}\hat{a}}$,
- $[\delta_C, \nabla^C_{\hat{a}}] = \nabla^C_{\delta_{\hat{A}}\hat{a}}$,
- $\partial \circ \delta_C = \delta_E \circ \partial$

for all $\hat{a} \in \Gamma(\hat{A})$. Correspondence $X \mapsto (\delta_{\hat{A}}, \delta_C, \delta_E)$ is 1-to-1.
Consider a VB groupoid and its VB algebroid

\[ \Omega \xrightarrow{\delta} E \] \[ G \xrightarrow{\delta} M \]

and

\[ W \xrightarrow{\delta} E \] \[ A \xrightarrow{\delta} M \]

**Theorem**

The Lie functor maps \( \text{aut}(\Omega, E \parallel G, M) \) to \( \text{aut}(W, E \parallel A, M) \).

If \( \Omega \) is s-simply-connected we get 1-to-1 correspondences:

\[ \text{aut}(\Omega, E \parallel G, M) \equiv \text{aut}(W, E \parallel A, M) \]

\[ \{ \text{multiplicative derivations of } \Omega \to G \} \equiv \{ \text{IM derivations of } W \to A \} \]

\[ \cong \{ \text{Algebraic data } (\delta_A, \delta_E, \delta_C) \} \]

Algebraic data corresponding to \( X \in \text{aut}(\Omega, E \parallel G, M) \) are given by:

\[ \delta_A \hat{\alpha} = [X, \hat{\alpha}], \quad \delta_E = \delta_{\tilde{s}*}X = \delta_{\tilde{t}*}X, \quad \delta_C = \delta_X|_C. \]
1. Multiplicative Vector Fields
2. VB Groupoids and Algebroids
3. Multiplicative Derivations
4. Morphic Derivations
5. Deformation Complexes
Can multiplicative derivations be described as groupoid maps?

Consider a VB groupoid: \((\Omega, E \rightharpoonup \mathcal{G}, M)\).

**Remark**

A vector field \(X \in \mathfrak{X}(\Omega)\) is an infinitesimal automorphism iff it is a VB groupoid map

\[ X : (\Omega, E \rightharpoonup \mathcal{G}, M) \to (T\Omega, TE \rightharpoonup T\mathcal{G}, TM). \]

Similarly for VB algebroids. This is partly unsatisfactory:

*we would like to think directly in terms of derivation \(\delta_X : \Gamma(\Omega) \to \Gamma(\Omega)!\)*

**Remark**

Derivations of a vector bundle \(E \to M\) are sections of a VB (actually a Lie algebroid): The *gauge algebroid* \(DE \to M\).
Let \((Ω, E \rightharpoonup G, M)\) be a VB groupoid.

**Remark**

A multiplicative derivation \(δ = δ_X\) is a section of the gauge algebroid \(DΩ → G\). Unlike \(TG\), the gauge algebroid \(DΩ\) is not a groupoid, in general. So, it doesn’t make sense to declare that \(δ : G → DΩ\) is morphic (i.e. a groupoid map).

**Proposition**

If \(C = 0\), then \(G\) acts on \(E\), and \(Ω ∼ s^*E \rightharpoonup E\) is the action groupoid.

**Theorem**

Let \(C = 0\), \(A\) be the Lie algebroid of \(G\), and \(δ\) a derivation of \(s^*E \rightarrow G\). Then
1. \((D(s^*E), DE \rightharpoonup G, M)\) is a VB groupoid,
2. \((D(A ×_M E), DE \rightharpoonup A, M)\) is its VB algebroid,
3. \(δ\) is multiplicative iff \(δ : G → D(s^*E)\) is morphic,
4. like-wise for a derivation of \(A ×_M E → A\).
Multiplicative Vector Fields

VB Groupoids and Algebroids

Multiplicative Derivations

Morphic Derivations

Deformation Complexes
Deformation Complex of a Lie Groupoid

Let \( G \to M \) be a Lie groupoid with Lie algebroid \( A \to M \).

Denote by \( \bar{m} : G \times_{s,s} G \to G, (g, h) \mapsto gh^{-1} \) the division.

**Definition [CRAINIC, MESTRE, STRUCHINER 2015]**

The deformation complex of \( G \) is \((C^\bullet_{\text{def}}(G), d_{\text{def}})\) where

1. \( C^k_{\text{def}}(G) \) consists of source-projectable maps \( G^{(k)} \to T G \), for \( k > 0 \),
2. \( C^0_{\text{def}}(G) = \Gamma(A) \),
3. \( d_{\text{def}} c(g_1, \ldots, g_{k+1}) = -d\bar{m}(c(g_1 g_2, g_3, \ldots, g_{k+1}), c(g_2, g_3, \ldots, g_{k+1})) \)
   \[ + \sum_{i=2}^{k} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) + (-1)^{k+1} c(g_1, \ldots, g_k), \]
   for all \( k > 0 \), and
4. \( d_{\text{def}} \alpha = \overleftarrow{\alpha} + \overrightarrow{\alpha} \), for \( \alpha \in \Gamma(A) \).

**Remark**

Multiplicative vector fields \( mod \) internal ones are \( H^1(C^\bullet_{\text{def}}(G), d_{\text{def}}) \).
Consider a VB groupoid $(\Omega, E \sslash G, M)$.

**Proposition/Definition**

The *linear deformation complex* of $(\Omega, E \sslash G, M)$ is the subcomplex

$$(C_{\text{def, lin}}(\Omega), d_{\text{def}}) \subset (C_{\text{def}}(\Omega), d_{\text{def}})$$

consisting of *linear cochains*, i.e. $\tilde{c} \in C_{\text{def}}(\Omega)$ which are VB maps:

$$\Omega(\bullet) \xrightarrow{\tilde{c}} T\Omega$$

$$G(\bullet) \xrightarrow{c} TG$$

**Remark**

Infinitesimal automorphisms of $(\Omega, E \sslash G, M)$ *mod* internal ones are $H^1(C_{\text{def, lin}}(\Omega), d_{\text{def}})$. 
Deformation Complex of a Lie Algebroid

Let \( A \Rightarrow M \) be a Lie algebroid.

**Definition [Crainic, Moerdijk 2008]**

The deformation complex of \( A \) is \((C^\bullet_{\text{def}}(A), d_{\text{def}})\) where \( C^k_{\text{def}}(A) \) consists of \( k \)-multiderivations of \( A \) with multilinear symbol and

\[
d_{\text{def}}c(\alpha_1, \ldots, \alpha_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1}[\alpha_i, c(a_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1})]_A \\
+ \sum_{i<j} (-1)^{i+j}c([\alpha_i, \alpha_j]_A, \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}),
\]

\( \alpha_1, \ldots, \alpha_{k+1} \in \Gamma(A) \).

**Remark**

Lie algebroid derivations mod internal ones are \( H^1(C^\bullet_{\text{def}}(A), d_{\text{def}}) \).
Consider a VB algebroid $(W, E \parallel A, M)$.

**Proposition/Definition**

The *linear deformation complex* of $(W, E \parallel A, M)$ is the subcomplex

$$ (C_{\text{def, lin}}(W), d_{\text{def}}) \subset (C_{\text{def}}(W), d_{\text{def}}) $$

consisting of *linear cochains*, i.e. $\tilde{c} \in C_{\text{def}}(W)$ preserving linear sections.

**Remark**

Infinitesimal automorphisms of $(W, E \parallel A, M)$ *mod* internal ones are $H^1(C_{\text{def, lin}}(W), d_{\text{def}})$.

**Proposition (linear Van Est map)**

*There is a Van Est map:* $\text{VE} : (C_{\text{def}}(\Omega), d_{\text{def}}) \to (C_{\text{def}}(W), d_{\text{def}})$ restricting to linear deformations complexes.
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Thank you!