Products in Jacobi Geometry

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(based on joint work in progress with: J. SCHNITZER)

Products in Poisson Geometry

(M_1, ω_1) and (M_2, ω_2) symplectic manifolds
$\updownarrow$
$(M_1 \times M_2, \omega_1 + \omega_2)$ a symplectic manifold

Lagrangian calculus

(M_1, π_1) and (M_2, π_2) Poisson manifolds
$\updownarrow$
$(M_1 \times M_2, \pi_1 + \pi_2)$ a Poisson manifold

Coisotropic calculus

Products in Poisson Geometry

Weinstein Splitting

Every Poisson manifold (M, π) is locally of the form

$$\left(N \times O, \pi_N + \omega_O^{-1} \right)$$

with (N, π_N) having a 0-dimensional leaf.

Remarks

- $(M_1 \times M_2, \pi_1 + \pi_2)$ is *not* a categorical product;
- products of Dirac manifolds are Dirac;
- coisotropic/isotropic calculus for Dirac manifolds[UCHINO 2004];
- Splitting for Dirac manifolds[BLOHMANN 2017].

Products of Contact Manifolds?

Contact Geometry:

odd dimensional analogue of Symplectic Geometry.

Contact structure:

maximally non-integrable hyperplane distribution $H \subset TM$.
$\rightsquigarrow$ Contact manifolds are <u>odd-dimensional</u>.

> (M_1, H_1) and (M_2, H_2) contact manifolds, but...
>
> $M_1 \times M_2$ *cannot be a contact manifold*.

A solution within (Dirac-)Jacobi Geometry via *Homogenization*!

Outline

- Contact Geometry revisited

- A Symplectic-to-Contact dictionary

- Homogenization

- Products$^!$ of Dirac-Jacobi manifolds

- Products and splittings

Contact Geometry Revisited

Derivations

$E \to M$ a vector bundle.

Derivation of E:

$$\Delta : \Gamma(E) \to \Gamma(E)$$

- Δ is $\mathbb{R}$-linear,
- *Leibniz rule*: $\Delta(fe) = X(f)e + f\Delta(e)$.

$X =: \sigma(\Delta)$ is the *symbol*.

Remarks
- ▶ Derivations are sections of the *gauge algebroid*: $DE \Rightarrow M$;
- ▶ DE acts tautologically on E;
- ▶ the de Rham complex $(\Omega^\bullet(DE, E), d_{DE})$ is acyclic;
- ▶ $E = L$ a line bundle $\Rightarrow DL = \mathrm{Hom}(\mathfrak{J}^1 L, L)$ and $\mathfrak{J}^1 L = \mathrm{Hom}(DL, L)$.

Contact Geometry Revisited

Symplectic Atiyah Forms

(M, H) a contact manifold,
$L = TM/H$ the normal bundle,
$\theta : TM \to L$ the projection.

$$
\left\{ \begin{array}{c} \text{Contact structures } H \\ \text{with } TM/H = L \end{array} \right\} \quad \rightleftharpoons \quad \left\{ \begin{array}{c} \text{closed, non-degenerate} \\ \omega \in \Omega^2(DL, L) \end{array} \right\}
$$

$$
H \quad \mapsto \quad d_{DL}(\theta \circ \sigma)
$$

$$
\sigma(\ker \iota_{\mathbb{I}} \omega) \quad \leftarrow\!\shortmid \quad \omega
$$

Symplectic-to-Contact Dictionary:
- ▶ $C^\infty(M) \quad \rightsquigarrow \quad \Gamma(L)$
- ▶ $TM \quad \rightsquigarrow \quad DL$
- ▶ $T^*M \quad \rightsquigarrow \quad \mathfrak{J}^1 L$

Symplectic-to-Contact Dictionary

Jacobi Manifolds

Symplectic-to-Contact Dictionary:

▶ *Poisson ⇝ Jacobi*;
▶ *Dirac ⇝ Dirac-Jacobi*;
▶ *Generalized complex ⇝ Generalized contact*;

Jacobi manifold: manifold M with a line bundle $L \to M$ and

$$\{-, -\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L) \qquad (\textit{Jacobi bracket})$$

- a Lie bracket,
- a bi-derivation.

Symplectic-to-Contact Dictionary

Jacobi Geometry

A Jacobi bracket can be seen as $J : \wedge^2 \mathfrak{J}^1 L \to L$ (*Jacobi tensor*).

When $L = \mathbb{R}_M$ then $\mathfrak{J}^1 L = T^*M \oplus \mathbb{R}_M$ and

$$J \equiv (\Lambda, E) \in \mathfrak{X}^2(M) \oplus \mathfrak{X}(M)$$

$$J^\sharp : \mathfrak{J}^1 L \to DL$$

$\text{Im}(\sigma \circ J^\sharp) \subset TM$ integrable distribution $\rightsquigarrow$ characteristic foliation.

Remark

▶ Jacobi manifolds possess two kinds of leaves:
  odd-dimensional $\rightsquigarrow$ contact leaves;
  even-dimensional $\rightsquigarrow$ lcs leaves.

J invertible $\rightleftharpoons$ $\omega = J^{-1}$ symplectic Atiyah form $\rightleftharpoons$ H contact structure.

Dirac-Jacobi Manifolds

$\mathbb{T}M = TM \oplus T^*M \rightsquigarrow \mathbb{D}L = DL \oplus \mathfrak{J}^1 L$: *omni-Lie algebroid*[CHEN, LIU 2010]

Structures on $\mathbb{D}L$:

$\langle\!\langle (\Delta, \phi), (\nabla, \psi) \rangle\!\rangle = \langle \Delta, \psi \rangle + \langle \nabla, \phi \rangle,$

$[\![(\Delta, \phi), (\nabla, \psi)]\!] = ([\Delta, \nabla], \mathcal{L}_\Delta \psi - \iota_\nabla d_{DL} \phi).$

Dirac-Jacobi manifold:[WADE 2000][V 2018]
manifold M with a line bundle L and a subbundle $\mathfrak{L} \subset \mathbb{D}L$

- $\mathfrak{L}$ *maximally isotropic*,
- $\mathfrak{L}$ *involutive*.

Remarks

▶ There are backward and forward Dirac-Jacobi maps,
▶ Dirac-Jacobi manifolds possess two kinds of leaves:
   pre-contact leaves;
   lc pre-s leaves.

Examples

- $DL \subset \mathbb{D}L$ and $\mathfrak{J}^1 L \subset \mathbb{D}L$,
- Jacobi brackets $\{-,-\} \equiv J$ ($\mathfrak{L}_J = \mathrm{graph}(J^\sharp)$),
- pre-contact structures ($=$ hyperplane distributions),
- locally conformal pre-symplectic structures,
- Dirac structures,
- *homogeneous Poisson structures*.

Symplectic-to-Contact Dictionary

From Homogeneous Poisson to Dirac-Jacobi

Homogeneous Poisson manifold (Infinitesimal):

manifold P with (Z, π)

- Z a vector field such that,
- π a Poisson bivector

$$\mathcal{L}_Z \pi = -\pi$$

$$\left\{ \begin{array}{c} \text{Homogeneous Poisson} \\ \text{structures on } P \end{array} \right\} \quad \rightarrow \quad \left\{ \begin{array}{c} \text{Dirac-Jacobi} \\ \text{structures on } \mathbb{R}_P \end{array} \right\}^{\text{[WADE 2000]}}$$

$$(Z, \pi) \quad \mapsto \quad \mathfrak{L}_{(Z, \pi)}$$

$$\mathfrak{L}_{(Z,\pi)} = \left\{ \left(h - hZ + \pi^\sharp(\eta), \eta \otimes 1 + \eta(Z) \cdot j^1 1 \right) : h \in \mathbb{R}, \eta \in T^*P \right\}$$

From Contact to Homogeneous Symplectic

{Contact manifolds} $\leftrightharpoons$ {Homogeneous symplectic manifolds}

Homogeneous symplectic manifold (Finite):

manifold P with (h, ω)

- $h : \mathbb{R}^* \times P \to P$ a free and proper action,
- ω a symplectic form such that

$$h_r^*(\omega) = r \cdot \omega$$

$$
\begin{aligned}
\{\text{Line bundles}\} \quad &\leftrightharpoons \quad \{\text{Principal } \mathbb{R}^*\text{-bundles}\} \\
L \quad &\mapsto \quad \widetilde{L} = L^* \smallsetminus 0 \\
(P \times \mathbb{R})/\mathbb{R}^* \quad &\leftharpoondown \quad P
\end{aligned}
$$

Symplectization

$L \to M$ a line bundle,
$\widetilde{L} \xrightarrow{p} M$ the principal $\mathbb{R}^*$-bundle.

$$\begin{array}{ccc}
\mathbb{R}_{\widetilde{L}} \longrightarrow L & T\widetilde{L} \longrightarrow DL & T^*\widetilde{L} \longrightarrow \mathfrak{J}^1 L \\
\downarrow \qquad \downarrow & \downarrow \qquad \downarrow & \downarrow \qquad \downarrow \\
\widetilde{L} \xrightarrow{p} M & \widetilde{L} \xrightarrow{p} M & \widetilde{L} \xrightarrow{p} M
\end{array}$$

contact structure H with $TM/H = L$
$\updownarrow$
$\omega \in \Omega^2(DL, L)$ symplectic Atiyah 2-form
$\updownarrow$
$\widetilde{\omega} = p^*\omega$ homogeneous symplectic structure on $\widetilde{L}$

$\{$Contact manifolds$\}$ $\iff$ $\{$Homogeneous symplectic manifolds$\}$

(M, H) $\mapsto$ $(\widetilde{L}, \widetilde{\omega})$

Homogenization

Poisson-ization and Dirac-ization

$L \to M$ a line bundle,
$\widetilde{L} \overset{p}{\to} M$ the principal $\mathbb{R}^*$-bundle.

$$\{\text{Jacobi manifolds}\} \quad \leftrightharpoons \quad \{\text{Homogeneous Poisson manifolds}\}$$
$$(M, L, \{-, -\} \equiv J) \quad \mapsto \quad \left(\widetilde{L}, \widetilde{J} := p^* J\right)$$

$$\{\text{Dirac-Jacobi manifolds}\} \quad \leftrightharpoons \quad \{\text{Homogeneous Dirac manifolds}\}$$
$$(M, L, \mathfrak{L} \subset \mathbb{D}L) \quad \mapsto \quad \left(\widetilde{L}, \widetilde{\mathfrak{L}} := p^* \mathfrak{L} \subset \mathbb{T}\widetilde{L}\right)$$

Homogeneous Poisson structure (Finite): $h_r^*(\pi) = r^{-1} \cdot \pi$,

Homogeneous Dirac structure: $\left(Th_r \oplus r \cdot T^* h_{r^{-1}}\right)(\mathfrak{L}) = \mathfrak{L}$.

Products! of Line Bundles

$\mathbf{LB} = \{\text{line bundles}, \text{fiber-wise invertible line bundle maps}\}$.

Lemma

$\mathbf{LB}$ has products.

Proof. $L_1 \to M_1$, $L_2 \to M_2$ line bundles $\rightsquigarrow \widetilde{L}_1$, $\widetilde{L}_2$ principal $\mathbb{R}^*$-bundles.

$$\left(\widetilde{L}_1 \times \widetilde{L}_2, h \times h \right)$$

is a product in the category of principal $\mathbb{R}^*$-bundles.

$$\widetilde{L}_1 \times \widetilde{L}_2 \quad \rightsquigarrow \quad L_1 \times^! L_2 \to M_1 \times^! M_2$$

and $L_1 \times^! L_2$ is a product in $\mathbf{LB}$.

Products[!] of Line Bundles

▶ $M_1 \times^! M_2 = \mathsf{Iso}(L_1, L_2) = \left\{ \varphi : (L_1)_x \overset{\cong}{\to} (L_2)_y \mid (x, y) \in M_1 \times M_2 \right\},$

▶ $L_1 \times^! L_2 = \left(M_1 \times^! M_2 \right) \times_{M_1} L_1 \cong \left(M_1 \times^! M_2 \right) \times_{M_2} L_2.$

$$
\begin{array}{ccccc}
L_1 & \longleftarrow & L_1 \times^! L_2 & \longrightarrow & L_2 \\
\downarrow & & \downarrow & & \downarrow \\
M_1 & \longleftarrow & M_1 \times^! M_2 & \longrightarrow & M_2 \\
\| & & \downarrow & & \| \\
M_1 & \longleftarrow & M_1 \times M_2 & \longrightarrow & M_2
\end{array}
$$

$$
\left\{ \text{sections} \quad \begin{array}{c} M_1 \times^! M_2 \\ \downarrow \uparrow \sigma \\ M_1 \times M_2 \end{array} \right\}
\Longleftarrow
\left\{ \text{diagrams} \quad \begin{array}{ccc} L_1 \longleftarrow L \longrightarrow L_2 \\ \downarrow \quad \downarrow \quad \downarrow \\ M_1 \leftarrow M_1 \times M_2 \to M_2 \end{array} \right\}
$$

$$
\sigma \;\mapsto\; L = \sigma^* \left(L_1 \times^! L_2 \right)
$$

Products! of Dirac-Jacobi Manifolds

Lemma

(M_i, L_i) Dirac-Jacobi, $i = 1, 2 \implies (M_1 \times^! M_2, L_1 \times^! L_2)$ Dirac-Jacobi.

Proof.

$(M_1, L_1, \mathfrak{L}_1)$, $(M_2, L_2, \mathfrak{L}_2)$ Dirac-Jacobi manifolds

$\updownarrow$

$(\widetilde{L}_1, h, \widetilde{\mathfrak{L}}_1)$, $(\widetilde{L}_2, h, \widetilde{\mathfrak{L}}_2)$ homogeneous Dirac manifolds

$\updownarrow$

$\left(\widetilde{L}_1 \times \widetilde{L}_2, h \times h, \widetilde{\mathfrak{L}}_1 \times \widetilde{\mathfrak{L}}_2 \right)$

$\updownarrow$

$(M_1 \times^! M_2, L_1 \times^! L_2, \mathfrak{L}_1 \times^! \mathfrak{L}_2)$

The Product! Dirac-Jacobi Structure

$$\mathfrak{L}_1 \qquad \mathfrak{L}_1 \times^! \mathfrak{L}_2 = p_1^! \mathfrak{L}_1 \star p_2^! \mathfrak{L}_2 \qquad \mathfrak{L}_1$$

$$
\begin{array}{ccccc}
L_1 & \longleftarrow & L_1 \times^! L_2 & \longrightarrow & L_2 \\
\downarrow & & \llcorner \downarrow \lrcorner & & \downarrow \\
M_1 & \xleftarrow{\ p_1\ } & M_1 \times^! M_2 & \xrightarrow{\ p_2\ } & M_2 \\
\parallel & & \downarrow & & \parallel \\
M_1 & \longleftarrow & M_1 \times M_2 & \longrightarrow & M_2
\end{array}
$$

$p^! \mathfrak{L} = \{(\Delta, p^*\phi) : (p_*\Delta, \phi) \in \mathfrak{L}\},$

$\mathfrak{L} \star \mathfrak{L}' = \{(\Delta, \phi + \phi') : (\Delta, \phi) \in \mathfrak{L}, (\Delta, \phi') \in \mathfrak{L}'\}.$

Remarks

- contact $\times^!$ contact = contact,
- pre-contact $\times^!$ pre-contact = pre-contact,
- Jacobi $\times^!$ Jacobi = Jacobi.

Products and Splittings

Splittings of Jacobi Manifolds

$(M, L, \{-, -\} \equiv J)$ a Jacobi manifold.

Theorem[DAZORD, LICHNEROWICZ, MARLE 1991]

Let $x_0 \in M$ be a point in a **contact leaf** $\mathcal{O}$.

1. Every *small complementary transversal* $N \ni x_0$ possesses a homogeneous Poisson structure (Z_N, π_N) such that $Z_N|_{x_0} = \pi_N|_{x_0} = 0$;

2. There is a local Jacobi isomorphism

$$(M, L, J) \cong (N \times \mathbb{R}^{odd}, \mathbb{R}_{N \times \mathbb{R}^{odd}}, J^\times)$$

where $J^\times \equiv (\Lambda^\times, E^\times)$ and

$$\Lambda^\times = \Lambda_{can} + \pi_N + E_{can} \wedge Z_N \quad \text{and} \quad E^\times = E_{can}.$$

Products and Splittings

Splittings of Jacobi Manifolds

$(M, L, \{-, -\} \equiv J)$ a Jacobi manifold.

Theorem[DAZORD, LICHNEROWICZ, MARLE 1991]

Let $x_0 \in M$ be a point in a **lcs leaf** $\mathcal{O}$.

1. Every *small complementary transversal* $N \ni x_0$ possesses a Jacobi structure $J_N \equiv (\Lambda_N, E_N)$ such that $J_N|_{x_0} = 0$;

2. There is a local Jacobi isomorphism

$$(M, L, J) \cong (N \times \mathbb{R}^{even}, \mathbb{R}_{N \times \mathbb{R}^{even}}, J^\times)$$

where $J^\times \equiv (\Lambda^\times, E^\times)$ and

$$\Lambda^\times = \Lambda_N + \pi_{can} + E_N \wedge Z_{can} \quad \text{and} \quad E^\times = E_N.$$

Splittings of Jacobi Manifolds

The DLM splittings can be written in terms of products:

<u>contact leaf</u>: $\exists$ a section $\sigma : N \times \mathbb{R}^{odd} \to N \times^! \mathbb{R}^{odd}$

$$\mathbb{R}_{N \times \mathbb{R}^{odd}} = \sigma^* \left(\mathbb{R}_N \times^! \mathbb{R}_{\mathbb{R}^{odd}} \right)$$

$$\mathfrak{L}_{(\Lambda^\times, E^\times)} = \sigma^! \left(\mathfrak{L}_{(Z_N, \pi_N)} \times^! \mathfrak{L}_{(\Lambda_{can}, E_{can})} \right)$$

<u>lcs leaf</u>: $\exists$ a section $\sigma : N \times \mathbb{R}^{even} \to N \times^! \mathbb{R}^{even}$

$$\mathbb{R}_{N \times \mathbb{R}^{even}} = \sigma^* \left(\mathbb{R}_N \times^! \mathbb{R}_{\mathbb{R}^{even}} \right)$$

$$\mathfrak{L}_{(\Lambda^\times, E^\times)} = \sigma^! \left(\mathfrak{L}_{(\Lambda_N, E_N)} \times^! \mathfrak{L}_{(Z_{can}, \pi_{can})} \right)$$

Products and Splittings

Splittings of Dirac-Jacobi Manifolds

$(M, L, \mathfrak{L})$ a Dirac-Jacobi manifold.

Theorem[SCHNITZER 2019]

Let $x_0 \in M$ be a point in a **pre-contact leaf** $\mathcal{O}$.

1. Every *small complementary transversal* $N \ni x_0$ possesses a homogeneous Poisson structure (Z_N, π_N) such that $Z_N|_{x_0} = \pi_N|_{x_0} = 0$;

2. There is
   - a local section $\sigma : N \times \mathcal{O} \to N \times^! \mathcal{O}$ and
   - a local Dirac-Jacobi isomorphism up to a B-field

$$(M, L, \mathfrak{L}) \cong^B \left(N \times \mathcal{O}, \sigma^* \left(L|_N \times^! L|_\mathcal{O} \right), \sigma^! \left(\mathfrak{L}_{(Z_N, \pi_N)} \times^! DL|_\mathcal{O} \right) \right)$$

Products and Splittings

Splittings of Dirac-Jacobi Manifolds

$(M, L, \mathfrak{L})$ a Dirac-Jacobi manifold.

Theorem[SCHNITZER 2019]

Let $x_0 \in M$ be a point in a **lc pre-s leaf** $\mathcal{O}$.

1. Every *small complementary transversal* $N \ni x_0$ possesses a Jacobi structure J_N such that $J_N|_{x_0} = 0$;

2. There is
   - a local section $\sigma : N \times \mathcal{O} \to N \times^! \mathcal{O}$ and
   - a local Dirac-Jacobi isomorphism up to a B-field

$$(M, L, \mathfrak{L}) \cong^B \left(N \times \mathcal{O}, \sigma^* \left(L|_N \times^! L|_{\mathcal{O}} \right), \sigma^! \left(\mathfrak{L}_{J_N} \times^! DL|_{\mathcal{O}} \right) \right)$$

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Thank you!