

HOMOGENEOUS G -STRUCTURES

LUCA VITAGLIANO

ABSTRACT. The theory of G -structures provides a unified framework for a large class of geometric structures, including symplectic, complex and Riemannian structures, as well as foliations and many others. Surprisingly, contact geometry - the “odd-dimensional counterpart” of symplectic geometry - does not fit naturally into this picture. In this talk, after a quick review on the very basics of G -structures (with examples), I will introduce the notion of a homogeneous G -structure, which encompasses contact structures, as well as some other interesting examples appeared in the literature. These are the notes of a talk given at the Research Seminar of the Geometry Group of the DipMat, University of Salerno on June 06, 2019, and are based on a joint project with Alfonso Tortorella and Ori Yudilevich (KU Leuven).

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1. G-STRUCTURES

1.1. **Introduction.** The idea behind G -structures is that

- (1) a geometric structure is determined by its space of compatible frames;
- (2) one can pass from a compatible frame to another one by acting via a suitable subgroup $G \subset \text{GL}(n, \mathbb{R})$.

Example 1.1.

- A symplectic structure is determined by its symplectic frames, and one can pass from a symplectic frame to another one by acting via $\text{Sp}(2k)$;
- A Riemannian structure is determined by its orthonormal frames, and one can pass from an orthonormal frame to another one by acting via $\text{O}(n)$;
- A complex structure is determined by its complex frames, and one can pass from a complex frame to another one by acting via $\text{GL}(k, \mathbb{C})$;
- etc.

Let M be an n -dimensional manifold.

Definition 1.2. A *frame* at a point $x \in M$ is a linear isomorphism $\psi : \mathbb{R}^n \xrightarrow{\cong} T_x M$. The dual *coframe* is the induced isomorphism $\psi^* := (\psi^t)^{-1} : \mathbb{R}^n \xrightarrow{\cong} T_x^* M$. Equivalently, a frame at x is an ordered basis (v_1, \dots, v_n) in $T_x M$.

The space of all frames is a principal $\text{GL}(n, \mathbb{R})$ -bundle $\text{Fr}(M) \rightarrow M$ called the *frame bundle*. The (right) principal action is given by

$$\text{Fr}(M) \times \text{GL}(n, \mathbb{R}) \rightarrow \text{Fr}(M), \quad (\psi, A) \mapsto \psi \circ A.$$

Definition 1.3. A *field of frames* (or, simply, a *frame*) is a (local) section $\Psi : M \rightarrow \text{Fr}(M)$ of the frame bundle.

Example 1.4. Let (U, χ) be a chart. The coordinate vector fields

$$\left(\frac{\partial}{\partial \chi^1}, \dots, \frac{\partial}{\partial \chi^n} \right)$$

define a local frame denoted Ψ_χ and sometimes called a *holonomic frame*. The dual (field of) coframe(s) is

$$(d\chi^1, \dots, d\chi^n).$$

1.2. **Definition of G -Structures.** Let $G \subset \text{GL}(n, \mathbb{R})$ be a (closed) Lie subgroup.

Definition 1.5 (G -structure). A G -structure on M is a *reduction* to G of the structure group $\text{GL}(n, \mathbb{R})$ of the frame bundle $\text{Fr}(M)$, i.e. a subbundle $S \subset \text{Fr}(M)$ such that

- (1) S is invariant under the (restricted) G -action,
- (2) together with the G -action, $S \rightarrow M$ is a principal G -bundle.

In particular, any frame in S at a point $x \in M$ can be transformed into any other such frame by acting with an appropriate element in G .

Remark 1.6. It should be clear that a G -structure on M can be equivalently seen as a (global) section of the bundle $\text{Fr}(M)/G \rightarrow M$.

G -structures always exist locally. To see this, just notice that one can “generate a G -structure” from any field of frames Ψ just applying G to the values of Ψ at all points (in the domain of Ψ). But G -structures might not exist globally.

1.3. Examples.

Example 1.7 (Trivial examples). Let $G = \text{GL}(n, \mathbb{R})$. There exists exactly one $\text{GL}(n, \mathbb{R})$ -structure on M , which consists of the whole $\text{Fr}(M)$ and doesn’t give any additional structure on M .

At the other extreme, let $G = \{1\}$. A $\{1\}$ -structure on M is the (smooth) choice of a frame at each point of M . In other words, it is a global frame on M . Yet in other words, it is a *global parallelism* on M . Not all manifolds admit a global parallelism.

Example 1.8 (Almost symplectic structures). Let $n = 2k$ be even and let $G = \text{Sp}(2k)$ be the symplectic group. An $\text{Sp}(k)$ -structure S on M is the same as an *almost symplectic structure* on M , i.e. a non-degenerate 2-form ω on M . To see this, let ω be an almost symplectic structure. Then, the space $S \subset \text{Fr}(M)$ of point-wise symplectic frames wrt ω is an $\text{Sp}(k)$ -structure.

Conversely, let $S \subset \text{Fr}(M)$ be an $\text{Sp}(k)$ -structure, let Ψ be a local section of S , and let Ψ^* be the dual coframe. As we know, Ψ^* can be seen as a local (ordered) basis $(\Psi^1, \dots, \Psi^{2k})$ of 1-forms. Consider the local 2-form

$$\omega = \Psi^1 \wedge \Psi^{k+1} + \dots + \Psi^k \wedge \Psi^{2k}.$$

By construction ω is non-degenerate. Additionally, ω is locally independent of the choice of Ψ , because Ψ can only change by the point-wise action of $\text{Sp}(k)$. So all these ω patch to give a global non-degenerate 2-form.

Example 1.9 (Riemannian structures). Let $G = \text{O}(n)$ be the orthogonal group. An $\text{O}(n)$ -structure S on M is the same as a *Riemannian metric* g on M . To see this, let g be a Riemannian metric. Then, the space $S \subset \text{Fr}(M)$ of point-wise orthonormal frames wrt g is an $\text{O}(n)$ -structure.

Conversely, let $S \subset \text{Fr}(M)$ be an $\text{O}(n)$ -structure, let Ψ be a local section of S , and let $\Psi^* \equiv (\Psi^1, \dots, \Psi^n)$ be the dual coframe. Consider the Riemannian metric

$$g = \Psi^1 \odot \Psi^1 + \dots + \Psi^n \odot \Psi^n.$$

Clearly, g is locally independent of the choice of Ψ . So all these g patch to give a global Riemannian metric.

Example 1.10 (Almost complex structures). Let $n = 2k$ be even, and let $G = \text{GL}(k, \mathbb{C})$ be the complex group. Recall that $\text{GL}(k, \mathbb{C})$ can be seen as a closed Lie subgroup of $\text{GL}(2k, \mathbb{R})$ via the embedding

$$\text{GL}(k, \mathbb{C}) \hookrightarrow \text{GL}(2k, \mathbb{R}), \quad A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

A $\text{GL}(k, \mathbb{C})$ -structure S on M is the same as an *almost complex structure* J on M . To see this, let J be an almost complex structure. A real frame $\psi \equiv (\psi_1, \dots, \psi_{2k})$ at x is *complex*, if

$$J_x = \psi_1 \otimes \psi^{k+1} + \dots + \psi_k \otimes \psi^{2k} - \psi_{k+1} \otimes \psi^1 - \dots - \psi_{2k} \otimes \psi^k,$$

where $\psi^* \equiv (\psi^1, \dots, \psi^{2k})$ is the dual coframe. The space $S \subset \text{Fr}(M)$ of point-wise complex frames wrt J is a $\text{GL}(k, \mathbb{C})$ -structure.

Conversely, let $S \subset \text{Fr}(M)$ be a $\text{GL}(k, \mathbb{C})$ -structure, let $\Psi \equiv (\Psi_1, \dots, \Psi_{2k})$ be a local section of S , and let $\Psi^* \equiv (\Psi^1, \dots, \Psi^{2k})$ be the dual coframe. Consider the almost complex structure

$$J = \Psi_1 \otimes \Psi^{k+1} + \dots + \Psi_k \otimes \Psi^{2k} - \Psi_{k+1} \otimes \Psi^1 - \dots - \Psi_{2k} \otimes \Psi^k.$$

Clearly, J is locally independent of the choice of Ψ . So all these J patch to give an almost complex structure on M .

Example 1.11 (Distributions). Let $0 \leq k \leq n$ and let $G = \text{GL}(k, n-k; \mathbb{R}) \subset \text{GL}(n, \mathbb{R})$ be the subgroup consisting of $(k, n-k)$ block upper triangular matrices, i.e. matrices of the form

$$\begin{pmatrix} A & B \\ \mathbb{O} & C \end{pmatrix},$$

with $A \in \text{GL}(k, \mathbb{R})$, and $C \in \text{GL}(n-k, \mathbb{R})$. A $\text{GL}(k, n-k; \mathbb{R})$ -structure S on M is the same as a rank k distribution on M . To see this, let $D \subset TM$ be a rank k distribution. A frame $\psi \equiv (\psi_1, \dots, \psi_n)$ at x is *adapted* to D if

$$D_x = \langle \psi_1, \dots, \psi_k \rangle.$$

The space $S \subset \text{Fr}(M)$ of adapted frames is a $\text{GL}(k, n-k; \mathbb{R})$ -structure.

Conversely, let $S \subset \text{Fr}(M)$ be a $\text{GL}(k, n-k; \mathbb{R})$ -structure, and let $\Psi \equiv (\Psi_1, \dots, \Psi_n)$ be a local section of S . Consider the rank k distribution D spanned by the first k entries Ψ_1, \dots, Ψ_k of Ψ . Clearly, D is locally independent of the choice of Ψ . So all these D patch to give a rank k distribution on M .

Exercise 1.12. Discuss the cases when G is equal to the following:

$$\text{GL}^+(n, \mathbb{R}), \text{SL}(n, \mathbb{R}), \text{SO}(n), \text{U}(k), \text{SU}(k), \{(k, n-k) \text{ block diagonal matrices}\}, \dots$$

1.4. Integrability. In all examples above one can impose an *integrability condition* on the G -structure. Namely, an almost symplectic structure ω can be asked to be a honest symplectic structure, i.e. $d\omega = 0$. Similarly:

- a Riemannian metric g can be asked to be flat, i.e. $\text{Riem}(g) = 0$;
- an almost complex structure J can be asked to be a honest complex structure, i.e. $[J, J]^{FN} = 0$ (where $[-, -]^{FN}$ is the Frölicher-Nijenhuis bracket);
- a distribution D can be asked to be *involutive*, i.e. $[\Gamma(D), \Gamma(D)] \subset \Gamma(D)$;
- etc.

Actually there is a notion of *integrability* for a generic G -structure.

Definition 1.13. A G -structure S on M is *integrable* if, for any $x \in M$ there is a chart (U, χ) around x *adapted* to S , i.e. such that the coordinate frame

$$\Psi_\chi = \left(\frac{\partial}{\partial \chi^1}, \dots, \frac{\partial}{\partial \chi^n} \right)$$

takes values in S .

Example 1.14 (Symplectic structures). Let $n = 2k$ be even, let S be an $\mathrm{Sp}(k)$ -structure on M and let ω be the associated almost symplectic structure.

Theorem 1.15. The $\mathrm{Sp}(k)$ -structure S is integrable iff ω is a symplectic structure, i.e. $d\omega = 0$.

Proof. Let S be integrable, and let (U, χ) be an adapted chart around a given point $x \in M$. This means that

$$\omega = d\chi^1 \wedge d\chi^{k+1} + \dots + d\chi^k \wedge d\chi^{2k},$$

in U . In particular, $d\omega = 0$. The converse implication is the celebrated *Darboux Lemma*. \square

Example 1.16 (Flat Metrics). Let S be an $\mathrm{O}(n)$ -structure on M and let g be the associated Riemannian metric.

Theorem 1.17. The $\mathrm{O}(n)$ -structure S is integrable iff g is a flat metric.

Proof. Let S be integrable, and let (U, χ) be an adapted chart around a given point $x \in M$. This means that

$$g = d\chi^1 \odot d\chi^1 + \dots + d\chi^n \odot d\chi^n,$$

in U . In particular, g is flat. The converse implication is the classical fact that a flat Riemannian manifold is locally isometric to a Euclidean space. \square

Example 1.18 (Complex Structures). Let $n = 2k$ be even, let S be a $\mathrm{GL}(k, \mathbb{C})$ -structure on M and let J be the associated almost complex structure.

Theorem 1.19. The $\mathrm{GL}(k, \mathbb{C})$ -structure S is integrable iff J is a complex structure, i.e. $[J, J]^{FN} = 0$.

Proof. Let S be integrable, and let (U, χ) be an adapted chart around a given point $x \in M$. This means that

$$J = \frac{\partial}{\partial \chi^1} \otimes d\chi^{k+1} + \dots + \frac{\partial}{\partial \chi^k} \otimes d\chi^{2k} - \frac{\partial}{\partial \chi^{k+1}} \otimes d\chi^1 - \dots - \frac{\partial}{\partial \chi^{2k}} \otimes d\chi^k,$$

in U . In particular, J is a complex structure and $z^i := \chi^i + i\chi^{i+k}$ are holomorphic coordinates on U . The converse implication is the *Newlander-Nirenberg Theorem*. \square

Example 1.20 (Foliations). Let $0 \leq k \leq n$, let S be a $\mathrm{GL}(k, n-k; \mathbb{R})$ -structure on M and let D be the associated rank k distribution.

Theorem 1.21. The $\mathrm{GL}(k, n-k; \mathbb{R})$ -structure S is integrable iff D is an involutive distribution.

Proof. Let S be integrable, and let (U, χ) be an adapted chart around a given point $x \in M$. This means that

$$D = \left\langle \frac{\partial}{\partial \chi^1}, \dots, \frac{\partial}{\partial \chi^k} \right\rangle,$$

in U . In particular, D is involutive. The converse implication is the *Frobenius Theorem*. \square

The theory of *Obstructions to Integrability* of G -structures is a fascinating theory that originated several branches of current Geometry including the *Formal Theory of PDEs*, *Lie Pseudogroups* and *Deformation Theory*.

Exercise 1.22. Work out the cases when G is equal to the following:

$GL^+(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SO(n)$, $U(k)$, $SU(k)$, $\{(k, n - k) \text{ block diagonal matrices}\}$, \dots

what are the integrable G -structures in those cases?

2. CONTACT STRUCTURES AND SYMPLECTIZATION

2.1. Definition. *Contact Geometry* is an odd dimensional analogue of Symplectic Geometry.

Definition 2.1. A *contact structure* on a manifold M is a *maximally non-integrable* hyperplane distribution.

Let us explain this definition. Given a distribution $D \subset TM$ there is a canonical 2-form

$$\kappa_D : \wedge^2 D \rightarrow TM/D, \quad (X, Y) \mapsto [X, Y] \bmod D$$

called the *curvature* of D and measuring how far is D from being integrable. If κ_D has maximal rank, we say that D is maximally non-integrable. In the case of a hyperplane distribution H , the normal bundle $L := TM/H$ is a line bundle, hence H is maximally non-integrable iff the flat map

$$(\kappa_H)_b : H \rightarrow \text{Hom}(H, L)$$

is a vector bundle isomorphism.

Contact structures appear naturally in several situations including: the boundary of (some) symplectic manifolds, first and second order PDEs in one dependent variable, dissipative and time dependent mechanics, geometric quantization.

Contact Geometry is in many respect similar to Symplectic Geometry (the ultimate reason being the existence of a *symplectization trick*, see below). For instance we have

Theorem 2.2 (Darboux). *Let H be a contact structure on M . Then $\dim M$ is odd and, around every point of M , there are coordinates (x^i, u, p_i) such that, locally, H is the kernel of the 1-form*

$$du - p_i dx^i.$$

However, unlike symplectic structures, contact structures are not integrable G -structures. As they are hyperplane distributions, of course they are $GL(n - 1, 1; \mathbb{R})$ -structures, but *not* integrable ones.

2.2. Symplectization. On the other hand a contact structure can be seen as a symplectic structure on a larger space. To see this, first consider an arbitrary line bundle $L \rightarrow M$, and let $\tilde{L} = L^* \setminus 0$. Fiber-wise scalar multiplication by a non-zero real number turns \tilde{L} into a principal \mathbb{R}^* -bundle (here \mathbb{R}^* is the multiplicative (Lie) group of non-zero reals) and every principal \mathbb{R}^* -bundle arises in this way. We will denote by

$$h : \mathbb{R}^* \times \tilde{L} \rightarrow \tilde{L}, \quad (r, \epsilon) \mapsto h_r(\epsilon) := r \cdot \epsilon,$$

the (principal) action.

Theorem 2.3. *There is a canonical one-to-one correspondence (up to isomorphisms) between:*

- *principal \mathbb{R}^* -bundles $\tilde{L} \rightarrow M$, equipped with a homogeneous symplectic structure on \tilde{L} , i.e. a symplectic structure ω such that*

$$h_r^* \omega = r \cdot \omega, \quad \text{for all } r \in \mathbb{R}^*, \quad (1)$$

and

- *contact structures on M .*

Proof (a sketch). Let $H \subset TM$ be a contact structure and let $L = TM/H$ be its normal bundle. The projection $TM \rightarrow TM/H$ can be seen as an L -valued 1-form $\theta \in \Omega^1(M, L)$. As a section of L is the same as a fiber-wise linear function on L^* , the 1-form θ can be pulled-back to a 1-form $\Theta \in \Omega^1(\tilde{L})$. Put

$$\omega = d\Theta.$$

It is easy to see that ω satisfies the homogeneity condition (1). Additionally, H being maximally non-integrable is equivalent to ω being non-degenerate.

Conversely, let ω be a homogeneous symplectic structure on the total space of some \mathbb{R}^* -principal bundle \tilde{L} . Notice that \tilde{L} is canonically equipped with a vertical vector field \mathcal{E} : the *Euler vector field*, i.e. the fundamental vector field associated to the canonical element 1 in the Lie algebra \mathbb{R} of the structure group \mathbb{R}^* . Put

$$\Theta = i_{\mathcal{E}} \omega.$$

From the homogeneity condition, Θ descends to an L -valued 1-form $\theta : TM \rightarrow L$ on M . Put $H = \ker \theta$. It can be checked that H is a contact structure. \square

Remark 2.4. There is a much more conceptual proof that I omit for space reasons.

Remark 2.5. More precisely, one can prove that there is an equivalence of categories between the category of contact manifolds, and the category of *homogeneous symplectic manifolds*.

By definition, the (homogeneous) symplectic manifold (\tilde{L}, ω) is the *symplectization* of the contact manifold (M, H) .

2.3. Contact Darboux Frames and Homogeneous Symplectic Frames. So contact structures on a $(2k + 1)$ -dimensional manifold are equivalent to homogeneous symplectic structures on a $(2k + 2)$ -dimensional principal \mathbb{R}^* -bundle. In their turn symplectic structures are (integrable) $\mathrm{Sp}(k + 1)$ -structures. In order to describe contact structures as G -structures, it then remains to describe the homogeneity condition on a symplectic structure purely in terms of the associated $\mathrm{Sp}(k + 1)$ -structure. To do this begin with a contact structure $H \subset TM$ on an n -dimensional manifold M , with $n = 2k + 1$, and let (\tilde{L}, ω) be the associated homogeneous symplectic manifold. Choose *Darboux coordinates* (x^i, u, p_i) for H , and denote $\lambda = \partial/\partial u \bmod H \in \Gamma(L)$. As we already mentioned, λ can be interpreted as a function on \tilde{L} . Locally, we have

$$\omega = d\lambda \wedge du - d(\lambda p_i) \wedge dx^i,$$

showing that

$$\Psi = \left(\frac{\partial}{\partial \lambda}, -\frac{\partial}{\partial \lambda p_i}, \frac{\partial}{\partial u}, \frac{\partial}{\partial x^i} \right)$$

is a symplectic frame. Additionally Ψ has the following *homogeneity property*:

$$d_\epsilon h_r \circ \Psi_\epsilon = \Psi_{r \cdot \epsilon} \circ A(r), \quad \text{for all } r \in \mathbb{R}^*,$$

where

$$A(r) = \begin{pmatrix} r\mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix}.$$

In other words the following diagram

$$\begin{array}{ccc} \mathbb{R}^{n+1} & \xrightarrow{\Psi_{r \cdot \epsilon}} & T_{r \cdot \epsilon} \tilde{L} \\ A(r) \uparrow & & \uparrow d_\epsilon h_r \\ \mathbb{R}^{n+1} & \xrightarrow{\Psi_\epsilon} & T_\epsilon \tilde{L} \end{array}$$

commutes.

Notice that $A : \mathbb{R}^* \rightarrow \mathrm{GL}(n + 1, \mathbb{R})$ is a Lie group homomorphism and the homogeneity condition on Ψ can be read as invariance under the following action of \mathbb{R}^* on the frame bundle $\mathrm{Fr}(\tilde{L})$ of \tilde{L} :

$$h^A : \mathbb{R} \times \mathrm{Fr}(\tilde{L}) \rightarrow \mathrm{Fr}(\tilde{L}), \quad (r, \psi) \mapsto h_r^A(\psi) := dh_r \circ \psi \circ A(r)^{-1}.$$

If we now denote by S the $\mathrm{Sp}(k + 1)$ -structure associated to ω , then

- (1) locally S possesses a section Ψ with an appropriate homogeneity property...
- (2) ... written in terms of a certain Lie group homomorphism $A : \mathbb{R}^* \rightarrow \mathrm{GL}(n + 1, \mathbb{R})$.

Two more things can be easily checked:

- (3) the homomorphism A takes values in the normalizer of $\mathrm{Sp}(k + 1)$ in $\mathrm{GL}(n + 1, \mathbb{R})$,
- (4) S is invariant under the action h^A of \mathbb{R}^* on $\mathrm{Fr}(\tilde{L})$ induced by A .

The main idea behind the *homogeneous G -structures* of the title is axiomatizing properties (1)–(4).

3. HOMOGENEOUS G-STRUCTURES

3.1. Homogeneous Frames: Degree. Our next aim is axiomatizing properties (1)–(4) at the end of the previous section. We begin defining what does it mean that a field of frames in homogeneous. In the following I will be rather sloppy about various *local vs global* and *connectedness vs non-connectedness* issues. but I guarantee that everything can be made very precise. So, take a principal \mathbb{R}^* -bundle $\tilde{L} \rightarrow M$ on an n -dimensional manifold, and consider the frame bundle $\text{Fr}(\tilde{L})$ of the total space \tilde{L} . Given a field of frames Ψ on \tilde{L} , a point $\epsilon \in \tilde{L}$ and a non-zero real number $r \in \mathbb{R}^*$, there is a unique invertible matrix $A_\Psi(\epsilon, r) \in \text{GL}(n+1, \mathbb{R})$ such that

$$d_\epsilon h_r \circ \Psi_\epsilon = \Psi_{r \cdot \epsilon} \circ A_\Psi(\epsilon, r).$$

Clearly

$$A_\Psi : \tilde{L} \times \mathbb{R}^* \rightarrow \text{GL}(n+1, \mathbb{R}), \quad (\epsilon, r) \mapsto A_\Psi(\epsilon, r)$$

is a smooth map.

Definition 3.1. The frame Ψ is *homogeneous* if the smooth function $A_\Psi = A_\Psi(\epsilon, r)$ is independent of ϵ .

Proposition 3.2. *Let Ψ be a homogeneous frame. Then $A_\Psi : \mathbb{R}^* \rightarrow \text{GL}(n+1, \mathbb{R})$ is a Lie group homomorphism (and Ψ is invariant under the induced action h^{A_Ψ} on $\text{Fr}(\tilde{L})$).*

Let Ψ be a homogeneous frame.

Definition 3.3. The Lie group homomorphism A_Ψ is called the *degree* of Ψ .

It is easy to construct local homogeneous frames of any arbitrary degree.

3.2. HGS: Definition. We begin with a

Proposition 3.4. *Let $S \subset \text{Fr}(\tilde{L})$ be a G-structure. Suppose that S possesses a homogeneous frame Ψ . Then the following two conditions are equivalent:*

- $A_\Psi : \mathbb{R}^* \rightarrow \text{GL}(n+1, \mathbb{R})$ takes values in the normalizer $N(G)$ of G in $\text{GL}(n+1, \mathbb{R})$;
- S is preserved by the action $h^{A_\Psi} : \mathbb{R}^* \times \text{Fr}(\tilde{L}) \rightarrow \text{Fr}(\tilde{L})$.

Definition 3.5. A G-structure S on \tilde{L} is *homogeneous* if locally, around every point, it possesses a homogeneous frame Ψ satisfying one (hence both) of the two equivalent conditions in Proposition 3.4.

3.3. HGS: Degree. Some remarks are in order. First of all, let S be a homogeneous G-structure, and let Ψ' be any homogeneous frame of S . Then automatically the degree $A_{\Psi'}$ of Ψ' takes values in the normalizer $N(G)$. Second, a homogeneous G-structure might possess two homogeneous frames Ψ, Ψ' of different degrees A, A' . However, A and A' cannot be that different. We actually have

$$A(r)G = A'(r)G, \quad \text{for all } r \in \mathbb{R}^*.$$

This shows that the composition

$$\mathbb{R}^* \xrightarrow{A_\Psi} N(G) \longrightarrow N(G)/G$$

is independent of the choice of a homogeneous frame Ψ of S , and it is rather intrinsic to S . We call it the *degree of S* , and denote it by

$$\alpha_S : \mathbb{R}^* \rightarrow N(G)/G.$$

3.4. HGS: Homogeneous Integrability. We now come to integrability. For homogeneous G -structures, we need an *adapted* notion of integrability. We begin defining what does it mean for a chart (U, χ) on \tilde{L} to be homogeneous.

Definition 3.6. A chart (U, χ) on \tilde{L} is *homogeneous* (of degree A) if the coordinate frame

$$\Psi_\chi \equiv \left(\frac{\partial}{\partial \chi^1}, \dots, \frac{\partial}{\partial \chi^{n+1}} \right)$$

is homogeneous (of degree A).

Proposition 3.7. A chart (U, χ) on \tilde{L} is homogeneous iff there exists a Lie group homomorphism

$$(A, b) : \mathbb{R}^* \rightarrow \text{Aff}(n+1, \mathbb{R}) = \text{GL}(n+1, \mathbb{R}) \ltimes \mathbb{R}, \quad r \mapsto (A(r), b(r))$$

such that

$$h_r^* \chi = A(r) \cdot \chi + b(r), \quad \text{for all } r \in \mathbb{R}^*,$$

and, in this case, (U, χ) is homogeneous of degree A .

Definition 3.8. A homogeneous G -structure S is *homogeneous integrable* if locally, around every point, there exists a homogeneous chart (U, χ) adapted to S , i.e. such that the coordinate frame Ψ_χ takes values in S .

Clearly, every homogeneous integrable homogeneous G -structure is also integrable in the usual sense. As I will discuss below, in all the examples we studied, the converse is also true. We are thus led to the following

Conjecture 3.9. (In all relevant cases) A homogeneous G -structure is homogeneous integrable iff it is integrable in the usual sense.

4. EXAMPLES

In this section we work out three examples for $G = \text{Sp}(k+1), \text{O}(n+1), \text{GL}(k+1, \mathbb{C})$. The idea will be always the same:

- (1) we compute the normalizer $N(G)$, and the quotient $N(G)/G$;
- (2) we pick up a Lie group homomorphism $\alpha : \mathbb{R}^* \rightarrow N(G)/G$;
- (3) we describe homogeneous G -structures of degree α ;
- (4) we discuss homogeneous integrability.

Contact structures are equivalent to homogeneous integrable homogeneous $\mathrm{Sp}(k)$ -structures of an appropriate degree (to be discussed below). In the following $\tilde{L} \rightarrow M$ will always be a principal \mathbb{R}^* -bundle over an n -dimensional manifold.

4.1. (Almost) Contact Structures. Let $n = 2k + 1$, and $G = \mathrm{Sp}(k + 1)$.

Proposition 4.1. *The normalizer $N(\mathrm{Sp}(k + 1))$ of the symplectic group in $\mathrm{GL}(n + 1, \mathbb{R})$ sits in a short exact sequence*

$$1 \rightarrow \mathrm{Sp}(k + 1) \rightarrow N(\mathrm{Sp}(k + 1)) \rightarrow \mathbb{R}^* \rightarrow 1$$

that splits via

$$A : \mathbb{R}^* \rightarrow N(\mathrm{Sp}(k + 1)), \quad r \mapsto \begin{pmatrix} r\mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix}.$$

In particular $N(\mathrm{Sp}(k + 1))/\mathrm{Sp}(k + 1) = \mathbb{R}^*$

In view of the discussion in Section 2, Proposition 4.1 already shows that contact structures are equivalent to integrable homogeneous $\mathrm{Sp}(k)$ -structures of degree $\alpha : \mathbb{R}^* \rightarrow N(\mathrm{Sp}(k + 1))/\mathrm{Sp}(k + 1) = \mathbb{R}^*$ given by the identical homomorphism $\mathrm{id}_{\mathbb{R}^*}$. More precisely, we have the following

Theorem 4.2. *There is a canonical one-to-one correspondence (up to isomorphisms) between:*

- (i) *principal \mathbb{R}^* bundles $\tilde{L} \rightarrow M$ equipped with a homogeneous $\mathrm{Sp}(k + 1)$ -structure S of degree $\mathrm{id}_{\mathbb{R}^*}$, and*
- (ii) *manifolds M equipped with a pair (H, Υ) consisting of*
 - *an hyperplane distribution $H \subset TM$, and*
 - *a 2-form Υ on M with values in the normal bundle $L := TM/H$, such that*

$$\lrcorner_H - \Upsilon|_H$$

is a non-degenerate L -valued 2-form on H .

Additionally, the following conditions are equivalent

- (1) *S is homogeneous integrable,*
- (2) *S is integrable,*
- (3) *$\Upsilon = 0$ (in particular H is a contact structure).*

According to this theorem it would be reasonable to call a pair (H, Υ) like in the statement an *almost contact structure* on M . Unfortunately, this terminology has been already taken for a completely different class of structures (that have little to do with contact structures), see below.

4.2. Atiyah Riemannian Structures. Let $G = \mathrm{O}(n + 1)$.

Proposition 4.3. *The normalizer $N(\mathrm{O}(n + 1))$ of the orthogonal group sits in a short exact sequence*

$$1 \rightarrow \mathrm{O}(n + 1) \rightarrow N(\mathrm{O}(n + 1)) \rightarrow \mathbb{R}_{>0}^* \rightarrow 1$$

that splits via

$$B : \mathbb{R}_{>0}^* \rightarrow N(\mathrm{O}(n+1)), \quad r \mapsto \sqrt{r} \cdot \mathbb{I}.$$

In particular $N(\mathrm{O}(n+1))/\mathrm{O}(n+1) = \mathbb{R}_{>0}^*$ (the multiplicative group of positive reals).

In the next theorem we describe (integrable) homogeneous $\mathrm{O}(n+1)$ -structures of degree $\alpha : \mathbb{R}^* \rightarrow N(\mathrm{O}(n+1))/\mathrm{O}(n+1) = \mathbb{R}_{>0}^*$ given by $\alpha(r) = \sqrt{|r|}$: the square-root of the absolute value.

Theorem 4.4. *There is a canonical one-to-one correspondence (up to isomorphisms) between:*

- (i) *principal \mathbb{R}^* bundles $\tilde{L} \rightarrow M$ equipped with a homogeneous $\mathrm{O}(n+1)$ -structure S of degree equal to the square root of the absolute value, and*
- (ii) *manifolds M equipped with a 4-tuples (L, ϕ, g, η) consisting of*
 - *a line bundle $L \rightarrow M$,*
 - *a global trivialization $\phi : |L| \cong M \times \mathbb{R}$,*
 - *a Riemannian metric g on M ,*
 - *a 1-form $\eta \in \Omega^1(M)$,**with no further restrictions.*

Additionally, the following conditions are equivalent

- (1) *S is homogeneous integrable,*
- (2) *S is integrable,*
- (3) *$\eta = 0$, and g has constant curvature equal to $1/4$.*

We call a 4-tuple like in the statement a (*flat*) *Atiyah Riemannian metric*.

4.3. Atiyah (Almost) Complex Structures. Let $n = 2k + 1$, and $G = \mathrm{GL}(k + 1, \mathbb{C})$.

Proposition 4.5. *The normalizer $N(\mathrm{GL}(k + 1, \mathbb{C}))$ of the complex group in $\mathrm{GL}(n + 1, \mathbb{R})$ sits in a short exact sequence*

$$1 \rightarrow \mathrm{GL}(k + 1, \mathbb{C}) \rightarrow N(\mathrm{GL}(k + 1, \mathbb{C})) \rightarrow \mathbb{Z}_2 \rightarrow 1$$

that splits via

$$C : \mathbb{R}^* \rightarrow N(\mathrm{GL}(k + 1, \mathbb{C})), \quad r \mapsto \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}.$$

In particular $N(\mathrm{GL}(k + 1, \mathbb{C}))/\mathrm{GL}(k + 1, \mathbb{C}) = \mathbb{Z}_2$

In order to describe homogeneous $\mathrm{GL}(k + 1, \mathbb{C})$ structures in an efficient way, we need to discuss the gauge algebroid of a vector bundle first. So, let $E \rightarrow M$ be a vector bundle. A *derivation* of E is an \mathbb{R} -linear operator

$$\Delta : \Gamma(E) \rightarrow \Gamma(E)$$

satisfying the following *Leibniz rule*:

$$\Delta(fe) = f\Delta(e) + X(f)e, \quad \text{for all } f \in C^\infty(M), \text{ and } e \in \Gamma(E),$$

and a, necessarily unique, vector field X , called the *symbol* of Δ and denoted by $\sigma(\Delta)$. Derivations of E can be seen as infinitesimal automorphisms of E , and they are sections of a transitive Lie algebroid $DE \rightrightarrows M$ whose Lie bracket is the commutator of derivations, and whose anchor is the symbol $\Delta \mapsto \sigma(\Delta)$. The kernel of the symbol consists of endomorphisms of E covering the identity of M . Accordingly, the gauge algebroid sits in a short exact sequence, the *Atiyah sequence*:

$$0 \rightarrow \text{End}(E) \rightarrow DE \xrightarrow{\sigma} TM \rightarrow 0.$$

In the next theorem we describe (integrable) homogeneous $\text{GL}(k+1, \mathbb{C})$ -structures of degree $\alpha : \mathbb{R}^* \rightarrow N(\text{GL}(k+1, \mathbb{C}))/\text{GL}(k+1, \mathbb{C}) = \mathbb{Z}_2$ given by the trivial homomorphism $\bar{1}$.

Theorem 4.6. *There is a canonical one-to-one correspondence (up to isomorphisms) between:*

- (i) *principal \mathbb{R}^* bundles $\tilde{L} \rightarrow M$ equipped with a homogeneous $\text{GL}(k+1, \mathbb{C})$ -structure S of degree $\bar{1}$, and*
- (ii) *manifolds M equipped with a pair (L, K) consisting of*
 - *a line bundle $L \rightarrow M$*
 - *a fiber-wise complex structure K in the gauge algebroid $DL \rightarrow M$.*

Additionally, the following conditions are equivalent

- (1) *S is homogeneous integrable,*
- (2) *S is integrable,*
- (3) *K is a complex structure in the sense that its Lie algebroid Nijenhuis torsion vanishes identically.*

We call a pair (L, K) like in the statement an *(almost) Atiyah complex structure*. When L is the trivial line bundle $M \times \mathbb{R} \rightarrow M$, an almost Atiyah complex structure is basically equivalent to what people in Metric Contact Geometry call an *almost contact structure*. In their turn, integrable (almost) Atiyah complex structures (L, K) with $L = M \times \mathbb{R}$ are basically equivalent to what are called *normal almost contact structures*. There is an obvious crash in the terminology here.

DIPMAT, UNIVERSITÀ DEGLI STUDI DI SALERNO, VIA GIOVANNI PAOLO II N° 123, 84084 FISCIANO (SA) ITALY.
E-mail address: lvitagliano@unisa.it