The time-nonlocal harmonic oscillator and the time-nonlocal Ornstein Uhlenbeck process

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Based on an ongoing joint work with Nikolai Leonenko

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Outline

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$$E = \frac{p^2}{2m} + V \longrightarrow i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar}{2m}\Delta + V\right)\psi = \mathcal{H}\,\psi$$

We get the time-dependent classical Schröndinger equation.

Bound states and preservation of probability masses

Bound states: $\psi_n \in L^2$ s.t. $\mathcal{H}\,\psi_n = E_n \psi_n$ and $\|\psi_n\|^2_{L^2(\mathbb{R}^d)} = 1$.

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$$\int_{\mathbb{R}^3} |f_n(x,t)|^2\,dx = \int_{\mathbb{R}^3} |\psi_n(x)|^2\,dx = 1$$

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$$\frac{d^\alpha g}{dt^\alpha}(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} g'(\tau) dt =: \int_0^t \bar{\nu}_\alpha(t-\tau) g'(\tau) dt.$$

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Problem: is there preservation of probability masses? [N04], [AYH13]

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In general, we cannot use Wick rotations on these operator due to the *lack of scaling*.

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$\mathcal{U} : L^2(\varphi_0^2(x)dx) \to L^2(\mathbb{R}^d)$ is an isometry.

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For any $\mathbf{j} \in \mathbb{N}_0^d$, let $|\mathbf{j}| = j_1 + \cdots + j_d$ and

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Then we can define the d-dimensional Hermite polynomials as

$$H_{\mathbf{j}}(x) = \frac{(-1)^{|\mathbf{j}|}}{\varphi_0^2(x)} \frac{\partial^{\mathbf{j}} \varphi_0^2}{\partial x^{\mathbf{j}}}(x)$$

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Notice that if $f \in L^2(\mathbb{R}^d)$, then $f = \sum_{\mathbf{j}\in\mathbb{N}_0^d} f_{\mathbf{j}}\varphi_{\mathbf{j}}$. The power spectrum is defined as the sequence $(A_n(f))_{n\geq 0}$, where

$$A_n^2(f) = \sum_{|\mathbf{j}|=n} f_{\mathbf{j}}^2$$

Fractional Hermite-Sobolev spaces

For $s > 0$, we define the Hermite-Sobolev space $H^s_{\mathcal{H}}(\mathbb{R}^d)$ as the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ with respect to the norm

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- $H^2_{\mathcal{H}}(\mathbb{R}^d) = \mathrm{Dom}(\mathcal{H})$, where with $\mathcal{H}$ we mean the maximal self-adjoint extension of $\mathcal{H}$.

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- $H^2_{\mathcal{H}}(\mathbb{R}^d) = \mathrm{Dom}(\mathcal{H})$, where with $\mathcal{H}$ we mean the maximal self-adjoint extension of $\mathcal{H}$.
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Fractional Hermite-Sobolev spaces

For $s > 0$, we define the Hermite-Sobolev space $H^s_{\mathcal{H}}(\mathbb{R}^d)$ as the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ with respect to the norm

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- By Kato square-root theorem, $f \in H^1_{\mathcal{H}}(\mathbb{R}^d)$ if and only if $f \in H^1(\mathbb{R}^d) \cap L^2(V(x)\,dx)$.
- $\{H^s_{\mathcal{H}}(\mathbb{R}^d)\}_{s\in\mathbb{R}}$ is the Hilbert scale generated by $\sqrt{I+\mathcal{H}}$. Hence we can extend the operator $\mathcal{H} : H^{s+2}_{\mathcal{H}} \to H^s(\mathcal{H})$ by setting $(\mathcal{H}f)_{\mathbf{j}} = |\mathbf{j}|^2 f_{\mathbf{j}}$.

Outline

A stochastic counterpart

For a fixed Bernstein function Φ there exists a unique (in distribution) subordinator $\sigma_\Phi(t)$ such that

$$\mathbb{E}[e^{-\lambda\sigma_\Phi(t)}] = e^{-t\Phi(\lambda)}.$$

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Theorem

For any $\lambda \in \mathbb{C}$, the function $\mathfrak{e}_\Phi(t;\lambda) = \mathbb{E}[e^{\lambda L_\Phi(t)}]$ is well-defined and it is the unique solution of

$$\begin{cases} \dfrac{d^\Phi}{dt^\Phi}\mathfrak{e}_\Phi(t;\lambda) = \lambda\, \mathfrak{e}_\Phi(t;\lambda) & t > 0 \\ \mathfrak{e}_\Phi(0;\lambda) = 1. \end{cases}$$

Furthermore, if $\Phi(\lambda) = \lambda^\alpha$, then $\mathfrak{e}_\Phi(t;\lambda) = E_\alpha(\lambda t^\alpha)$, where E_α is the Mittag-Leffler function.

Strong and distributional solutions for the heat equation with potential

Theorem [AL,26?]

Let $g \in H^s_{\mathcal{H}}(\mathbb{R}^d)$ with *Fourier* coefficients $g_{\mathbf{j}}$. Then the Cauchy problem

$$\frac{\partial^\Phi u}{\partial t^\Phi}(t) = -\mathcal{H}\,u(t), \qquad u(0) = g$$

admits a unique solution $u \in C(\mathbb{R}^+_0; H^s_{\mathcal{H}}(\mathbb{R}^d)) \cap C(\mathbb{R}^+_0; H^{s+2}_{\mathcal{H}}(\mathbb{R}^d))$, with $\frac{\partial^\Phi u}{\partial t^\Phi} \in C(\mathbb{R}^+; H^s_{\mathcal{H}}(\mathbb{R}^d))$ given by

$$u(t) = \sum_{\mathbf{j}\in\mathbb{N}^d_0} \mathfrak{e}_\Phi(t; -|\mathbf{j}|) g_{\mathbf{j}} \varphi_{\mathbf{j}}.$$

In particular, if $s \geq -2$, $u(t) \in L^2(\mathbb{R}^d)$ for $t > 0$ and if $s > \frac{d}{2}$ the equation holds pointwise for all $t > 0$ and $x \in \mathbb{R}^d$.

Idea of the proof

Thanks to the intertwining relation, one can equivalently study the solutions of

$$\frac{\partial^{\Phi} v}{\partial t^{\Phi}} = Lv(t), \qquad v(0) = \varphi_0^{-1} g.$$

This is done by employing the same strategy as in [LMS, 13] and [ALP,21].

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Remark

We have a *partial parabolic smoothing*: we only move from $H^s_{\mathcal{H}}$ to $H^{s+2}_{\mathcal{H}}$ for positive time. This partial smoothing is sharp, see [AV,25].

Solutions to the naive time-nonlocal Schrödinger equation

Theorem [AL,26?]

Let $g \in H^s_{\mathcal{H}}(\mathbb{R}^d)$ with *Fourier* coefficients $g_{\mathbf{j}}$. Then the Cauchy problem

$$i\frac{\partial^\Phi u}{\partial t^\Phi}(t) = \mathcal{H}\, u(t), \qquad u(0) = g$$

admits a unique solution $u \in C(\mathbb{R}^+_0; H^s_{\mathcal{H}}(\mathbb{R}^d))$, with $\frac{\partial^\Phi u}{\partial t^\Phi} \in C(\mathbb{R}^+; H^{s-2}_{\mathcal{H}}(\mathbb{R}^d))$ given by

$$u(t) = \sum_{\mathbf{j}\in\mathbb{N}_0^d} \mathfrak{e}_\Phi(t; i|\mathbf{j}|) g_{\mathbf{j}} \varphi_{\mathbf{j}}.$$

In particular, if $s \geq 0$, $u(t) \in L^2(\mathbb{R}^d)$ for $t > 0$ and if $s > \frac{d}{2} + 2$ the equation holds pointwise for all $t > 0$ and $x \in \mathbb{R}^d$.

The fractional case: naive leads to stasis

Theorem [AL,26?]

Let $\Phi(\lambda) = \lambda^\alpha$ and $g \in H^s_{\mathcal{H}}(\mathbb{R}^d)$ with *Fourier* coefficients $g_{\mathbf{j}}$. Then the unique solution of the Cauchy problem

$$i\frac{\partial^\Phi u}{\partial t^\Phi}(t) = \mathcal{H}\, u(t), \qquad u(0) = g$$

belongs to $C(\mathbb{R}^+; H^{s+2}_{\mathcal{H}}(\mathbb{R}^d))$ and $\frac{\partial^\Phi u}{\partial t^\Phi} \in C(\mathbb{R}^+; H^s_{\mathcal{H}}(\mathbb{R}^d))$.
Furthermore

$$\lim_{t\to+\infty} \|u(t) - g_0\varphi_0\|_{H^{s+2}} = 0.$$

In particular, if $s > \frac{d}{2}$ the convergence is also locally uniform.

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In particular, if $s > \frac{d}{2}$ the convergence is also locally uniform.

The latter follows by the fact that $E_\alpha(int^\alpha) \to 0$ as $t \to +\infty$ for any $n \geq 1$. Hence the naive version of the time-fractional Schrödinger equation *kills* the typical oscillatory behaviour.

The time-fractional Schrödinger equation

Theorem [AL,26?]

Let $\Phi(\lambda) = \lambda^\alpha$ and $g \in H^s_{\mathcal{H}}(\mathbb{R}^d)$ with *Fourier* coefficients $g_{\mathbf{j}}$. Then the Cauchy problem

$$(-i)^{-\alpha}\frac{\partial^\Phi u}{\partial t^\Phi}(t) = \mathcal{H}\, u(t), \qquad u(0) = g$$

admits a unique solution $u \in C(\mathbb{R}_0^+; H^s_{\mathcal{H}}(\mathbb{R}^d))$, with $\frac{\partial^\Phi u}{\partial t^\Phi} \in C(\mathbb{R}^+; H^{s-2}_{\mathcal{H}}(\mathbb{R}^d))$, given by

$$u(t) = \sum_{\mathbf{j}\in\mathbb{N}_0^d} E_\alpha\left(|\mathbf{j}|(-it)^\alpha\right) g_{\mathbf{j}}\varphi_{\mathbf{j}}.$$

In particular, if $s \geq 0$, $u \in L^2(\mathbb{R}^d)$ and if $s > \frac{d}{2} + 2$ the equation holds also pointwise.

Convergence towards non-stable solutions

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Let $\Phi(\lambda) = \lambda^\alpha$ and $g \in H^s_{\mathcal{H}}(\mathbb{R}^d)$ with *Fourier* coefficients $g_{\mathbf{j}}$. Then the unique solution of the Cauchy problem

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satisfies

$$\lim_{t\to+\infty} \|u(t) - u_{\text{per}}(t)\|_{H^s_{\mathcal{H}}} = 0, \text{ where } u_{\text{per}}(t) = \sum_{\mathbf{j}\in\mathbb{N}_0^d} e^{-|\mathbf{j}|^{\frac{1}{\alpha}} it} g_{\mathbf{j}} \varphi_{\mathbf{j}}.$$

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The latter follows by the fact that [GKMR,20]

$$E_\alpha(n(-it)^\alpha) = e^{-n^{\frac{1}{\alpha}} it} + O(t^{-1}).$$

Remarks

- $\|u_{\text{per}}(t)\|^2_{L^2(\mathbb{R}^d)} = 1$: no asymptotic loss of probability mass.

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Theorem [AL, 26?]

Let

$$\mathcal{N}_0(g) = \{n = 1, 2, \cdots : g_{\mathbf{j}} = 0 \text{ for all } |\mathbf{j}| = n\} \cup \{0\}.$$

Then u_{per} is periodic if and only if one of the following two properties hold:

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- $\mathcal{N}_0(g) = \mathbb{N}_0 \setminus \{n\}$ for some $n \in \mathbb{N}$.
- For all $n_1, n_2 \notin \mathcal{N}_0(g)$ it holds $(n_1/n_2)^{1/\alpha} \in \mathbb{Q}$.

Outline

Inverse subordination of semigroups

Let $\{T_t\}_{t\geq 0}$ be an exponentially bounded semigroup on a Banach space $\mathbb{B}$, i.e., denoting by $\mathcal{L}(\mathbb{B};\mathbb{B})$ the space of bounded linear operators on $\mathbb{B}$, there exist two constants $C_1, C_2 > 0$ such that

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Then we can define, by means of Bochner subordination, the one-parameter family of bounded linear operators

$$T_t^{\Phi} = \mathbb{E}[T_{L_\Phi(t)}], \text{ that satisfies } \left\|T_t^{\Phi}\right\|_{\mathcal{L}(\mathbb{B};\mathbb{B})} \leq C_1\, \mathfrak{e}_\Phi(t; C_2).$$

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Let A be the generator of an exponentially bounded semigroup $\{T_t\}_{t\geq 0}$ on X and consider the one-parameter family of operators $\{T_t^\Phi\}_{t\geq 0}$. Then, for any $g \in \mathrm{Dom}(A)$ the function $u(t) = T_t^\Phi g$ is a strong solution of

$$\begin{cases} \dfrac{d^\Phi u}{dt^\Phi} = Au(t) & t > 0 \\ u(0) = g. \end{cases}$$

The time-changed Feynman-Kac semigroup

Let $\mathbf{W}$ be a Brownian motion (with variance $2t$) and consider the Feynman-Kac semigroup

$$T_t g(x) = \mathbb{E}_x \left[\exp\left(-\int_0^t V(\mathbf{W}(s))\, ds \right) g(\mathbf{W}(t)) \right].$$

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In $L^2(\mathbb{R}^d)$ it holds $T_t = e^{-t\,\mathcal{H}}$. Then if we consider an independent inverse subordinator L_Φ, we get

$$T_t^\Phi g(x) = \mathbb{E}_x \left[\exp\left(- \int_0^{L_\Phi(t)} V(\mathbf{W}(s))\, ds \right) g(\mathbf{W}(L_\Phi(t))) \right].$$

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T_t^Φ is related to the time-nonlocal Schrödinger equation?

The Feynman-Kac formula

Theorem [AL, 26]

For all $g \in L^2(\mathbb{R}^d)$, the function $u(t,x) = T_t^{\Phi} g(x)$ is the unique strong solution of

$$\frac{\partial^{\Phi} u}{\partial t^{\Phi}}(t) = -\mathcal{H}\, u(t), \qquad u(0) = g.$$

In particular, it holds

$$T_t^{\Phi} g = \sum_{\mathbf{j} \in \mathbb{N}_0^d} \mathfrak{e}_{\Phi}(t; -|\mathbf{j}|) g_{\mathbf{j}} \varphi_{\mathbf{j}}.$$

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To prove the statement, it is sufficient to observe that it is true for $g \in H^2_{\mathcal{H}}(\mathbb{R}^d)$, that is dense in $L^2(\mathbb{R}^d)$. Hence, by density we have the second part of the statement, that in turn implies the first part.

The time-nonlocal Ornstein-Uhlenbeck process

Now let us focus on L and let $\mathbf{U}$ be the solution of

$$d\mathbf{U}(t) = -\mathbf{U}(t)dt + \sqrt{2}d\mathbf{W}(t)$$

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Then we know that $P_t g(x) = \mathbb{E}_x[g(\mathbf{U}(t))]$, on $L^2(\varphi_0^2(x)\,dx)$, satisfies $P_t = e^{tL}$.

Theorem [AL, 26]

For all $g \in L^2(\varphi_0^2(x)\,dx)$, the function $u(t,x) = P_t^\Phi g(x)$ is the unique strong solution of

$$\frac{\partial^\Phi u}{\partial t^\Phi}(t) = Lu(t), \qquad u(0) = g.$$

In particular, it holds

$$P_t^\Phi g(x) = \sum_{\mathbf{j} \in \mathbb{N}_0^d} \mathfrak{e}_\Phi(t; -|\mathbf{j}|) g_{\mathbf{j}} Q_{\mathbf{j}}(x) = \mathbb{E}_x[g(\mathbf{U}(L_\Phi(t)))].$$

Integral kernels for P_t^Φ

Let us denote $\mathbf{U}^\Phi(t) := \mathbf{U}(L_\Phi(t))$, where L_Φ is independent of $\mathbf{U}$.

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Let us denote $\mathbf{U}^\Phi(t) := \mathbf{U}(L_\Phi(t))$, where L_Φ is independent of $\mathbf{U}$. Then if $p(t, y; x)$ is the pdf of $\mathbf{U}(t)$ with $\mathbf{U}(0) = x$, we have

$$\mathbb{P}_x(\mathbf{U}(L_\Phi(t)) \in B) = \int_B \mathbb{E}_x[p(L_\Phi(t), y; x)]\, dy =: \int_B p_\Phi(t, y; x)\, dy.$$

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On the other hand, for fixed $t > 0$, P_t^Φ is a bounded linear operator on $L^2(\varphi_0^2(x)\,dx)$, hence by Riesz representation theorem, there exists a kernel $\overline{p}_\Phi(t,y;x)$ such that

$$P_t^\Phi g(x) = \int_{\mathbb{R}^d} g(y)\overline{p}_\Phi(t,y;x)\varphi_0^2(y)\,dy.$$

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$$P_t^{\Phi} g(x) = \int_{\mathbb{R}^d} g(y) \overline{p}_{\Phi}(t, y; x) \varphi_0^2(y)\, dy.$$

Comparing the two kernels, we get the following equality:

$$\overline{p}_{\Phi}(t, y; x) = \frac{p(t, y; x)}{\varphi_0^2(y)}$$

Integral kernels for P_t^Φ

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We can now relate the time-changed OU process with T_t^Φ

Mehler's formula

Theorem [AL, 26?]

For any $g \in L^2(\mathbb{R}^d)$ it holds

$$T_t^{\Phi} g(x) = \int_{\mathbb{R}^d} g(y) k_{\Phi}(t, y; x)\, dy,$$

where

$$k_{\Phi}(t, y; x) = \mathbb{E}\left[\frac{1}{\left(2\pi\left(1 - e^{-2L_{\Phi}(t)}\right)\right)^{\frac{d}{2}}} \exp\left(-\frac{|y - e^{-L_{\Phi}(t)}x|^2}{2(1 - e^{-2L_{\Phi}(t)})} + \frac{|y|^2 - |x|^2}{4}\right)\right]$$

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To prove the statement, it is sufficient to observe that the intertwining relation between $-\mathcal{H}$ and L transfers to T_t^Φ and P_t^Φ, i.e. $T_t^\Phi = \mathcal{U}\, P_t^\Phi\, \mathcal{U}^{-1}$, that in turn implies

$$k_\Phi(t, y; x) = \varphi_0(y)\varphi_0(x)\overline{p}_\Phi(t, y; x) = \frac{\varphi_0(x)}{\varphi_0(y)} p_\Phi(t, y; x).$$

Further applications

- Despite exhibiting some apparently non-physical behaviour, the theory of time-fractional Schrödinger equations found use in quantum information theory, as in [ZGY, 21].

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- Actually, we didn't have time to discuss the main probabilistic application: if we *slightly change* the potential, considering $V(x) = \frac{|x|^2}{2}$, with the same strategy we can study some properties of a time-changed softly killed Brownian motion.

Outline

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Thank you for the attention!!!