A Stochastic Growth Model with Random Catastrophes Applied to Population Dynamics

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General Remarks

A **catastrophe** (or jump) is a randomly timed event that causes a discontinuity in the diffusion process $X(t)$, with $t \in I$, which restarts from a random point after each jump (Cf. [3]). The process evolves as a series of **cycles**, with each cycle having a random duration representing the interarrival time between consecutive catastrophes. The state space of $X(t)$ is $\mathbb{R}^+$.

Let Θ_n, for $n \in \mathbb{N}$, be the **arrival time** of the n-th catastrophe. The PDF of the absolutely continuous random variable Θ_n is denoted with $\gamma_n(t)$, i.e.

$$\gamma_n(t) := \frac{\mathsf{P}(\Theta_n \in \mathrm{d}t)}{\mathrm{d}t}, \qquad t \in I, \quad n \in \mathbb{N}.$$

Moreover, we assume that Θ_0 is **the initial time** $t_0 \geq 0$, i.e. $\mathsf{P}(\Theta_0 = t_0) = 1$. Furthermore, let I_n be the **random time between the n-th and the $(n+1)$-th catastrophes**, i.e. $I_n := \Theta_{n+1} - \Theta_n$, for any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Denoting by $\psi_n(t)$ the corresponding PDF, i.e.

$$\psi_n(t) := \frac{\mathsf{P}(I_n \in \mathrm{d}t)}{\mathrm{d}t}, \qquad t \in I, \quad n \in \mathbb{N}_0.$$

Let $\{X_n(t);\ t \in I\}_{n \in \mathbb{N}_0}$ be a sequence of diffusion processes with transition PDF given by

$$f_n(x, t \mid r, s)\mathrm{d}x := \mathrm{P}\left(X_n(t) \in \mathrm{d}x \mid X(s) = r\right), \quad t_0 \leq s < t,\ x, r \in \mathbb{R}^+,$$

for any $n \in \mathbb{N}_0$. Let $N(t)$ be the **counting process** that describes the number of jumps occurred until the time $t \in I$, i.e. $N(t) := \max\{n \in \mathbb{N} \mid \Theta_n \leq t\}$, $t \in I$. **The complete process** $X(t)$ is then defined as

$$X(t) := \sum_{n=0}^{+\infty} X_n(t)\, \mathbb{I}\left(N(t) = n\right), \qquad t \in I, \tag{1}$$

where $\mathbb{I}\left(A\right)$ denotes the indicator function, equal to 1 when A is true and 0 otherwise.

The Case Study

We will focus on the case in which $X_n(t)$ is a lognormal diffusion process having the following **infinitesimal moments**

$$A_1(x,t) := h_n^{\xi_1}(t)x, \qquad A_2(x) := \sigma^2 x^2,$$

where $h_n^{\xi_1}(t)$ is a positive and integrable function in any set (t_0, t) with $t \in I$ and $\xi_1 = (\eta, \beta^\intercal)^\intercal$. The process is the solution of the following SDE

$$\mathrm{d}X_n(t) = h_n^{\xi_1}(t)X_n(t)\mathrm{d}t + \sigma X_n(t)\mathrm{d}W(t), \tag{2}$$

where $W(t)$ denotes a standard Wiener process independent on the initial state X_0 for any $t \in I$. The corresponding expression of the resulting process is the following

$$X_n(t) = X_n(\theta_n)\exp\left(H_n^{\xi_2}(\theta_n, t) + \sigma\left(W(t) - W(\theta_n)\right)\right), \qquad t \geq \theta_n. \tag{3}$$

where, being $\eta, \sigma \in \mathbb{R}^+$, $Q_\beta(t) = \sum_{i=1}^{p} \beta_i t^i$, $P_\beta(t) = \frac{\mathrm{d}}{\mathrm{d}t} Q_\beta(t)$, $\xi_2 = (\theta^\intercal, \sigma^2)^\intercal$ and $p \geq 2$,

$$h_n^{\xi_1}(t) = \frac{P_\beta(t)e^{-Q_\beta(t)}}{\eta + e^{-Q_\beta(t)}}, \quad H_n^{\xi_2}(s,t) = \log\left(\frac{\eta + e^{-Q_\beta(s)}}{\eta + e^{-Q_\beta(t)}}\right) - \frac{\sigma^2}{2}(t-s), \quad s < t. \tag{4}$$

The mean of the process $X_n(t)$ is represented by a multi-sigmoidal logistic growth function, i.e.

$$\mathrm{E}\left(X_n(t)\right) := x(t) = x_0 \, \frac{\eta + e^{-Q_\beta(t_0)}}{\eta + e^{-Q_\beta(t)}}, \qquad t \in I, \tag{5}$$

being $x_0 := x(t_0) > 0$. The curve given in Eq. (5) is known as **multi-sigmoidal logistic function** since it may exhibit multiple inflections.
We will focus on the case in which $X_n(t)$ is a stochastic process having the following transition PDF, for any $n \in \mathbb{N}$ and for $t_0 < s < t,\ x, r_n \in \mathbb{R}^+$

$$f_n(x, t \mid r_n, s) = \frac{1}{x\sigma\sqrt{2\pi(t-s)}} \exp\left(-\frac{(\log x - \log r_n - H_n^{\xi_2}(s,t))^2}{2\sigma^2(t-s)}\right). \tag{6}$$

Deterministic Arrival Times

We consider the case where intertimes follow a degenerate distribution, i.e., $\mathrm{P}(I_n = \tau_n) = 1$ for all $n \in \mathbb{N}_0$ and the arrival time are defined as $\theta_n := \sum_{i=0}^{n-1} \tau_i$, where τ_i is the selected as the **first-crossing-time** of the deterministic growth curve $x_n(t) := E(X_n(t))$ through a fixed boundary B_n.

Strategies for B_n:

(B1) a **percentage** $p_C \in (0, 1)$ of the carrying capacity of $x_n(t)$;

(B2) a **multiple of the initial state of the growth curve**;

(B3) a **predetermined real value**;

(B4) a **time-dependent threshold** $S_n(t)$ defined as follows:

$$S_n(t) = \exp\left(\alpha_1 + \alpha_2(t - \theta_n) + H_n^{\xi_1}(\theta_n, t)\right), \quad t \geq \theta_n, \quad \alpha_1, \alpha_2 \in \mathbb{R}. \tag{7}$$

Random re-starting points

Regarding the choice of the distribution of the random state ρ_n, $n \in \mathbb{N}$, from which the process $X(t)$ restarts after the n-th catastrophe, let us focus on the following three possibilities:

(P1) **degenerate** re-starting point: ρ_n follows a degenerate distribution centered in $r_n \in \mathbb{R}^+$, i.e. $\mathrm{P}\left(\rho_n = r_n\right) = 1$;

(P2) **lognormal** random re-starting point: ρ_n follows a lognormal distribution, i.e. $\rho_n \sim \Lambda_1\left(\mu_n, \sigma_n\right)$, with $\mu_n, \sigma_n \in \mathbb{R}^+$;

(P3) **binomial** random re-starting point: ρ_n follows a binomial distribution, i.e. $\rho_n \sim \mathrm{Bin}(p_n, N_n)$, with $p_n \in (0,1)$ and $N_n \in \mathbb{N}$.

Note that cases (P1) and (P2) correspond to the usual choices made to study lognormal diffusion processes with a mean coinciding with deterministic growth curves, as seen, for example, in [2].

In [1] we performed an analysis of the case (P3). More in detail, after each catastrophe, the process restarts from $\rho_n \sim \text{Bin}(N_n, p_n)$, with $p_n \in (0,1)$ and $N_n \in \mathbb{N}$. The probability law of ρ_n is given by

$$P(\rho_n = r) = \binom{N_n}{r} p_n^r (1 - p_n)^{N_n - r}, \qquad n \in \mathbb{N},\ r = 0, 1, \ldots, N_n. \tag{8}$$

From Eq. (6), for $t \in [\theta_n, \theta_{n+1})$ we get, for any $x, x_0 \in \mathbb{R}^+$

$$f(x, t | x_0, t_0) = \sum_{r=0}^{N_n} \mathbb{P}(\rho_n = r) f_n(x, t | r, \theta_n). \tag{9}$$

We can refer to three different choices for the parameter N_n :

(N1) The **ceiling of the mean** of $[X(t) \mid X(\theta_{n-1})]$ for $t \to \theta_n$,

(N2) The **ceiling of the** α**-quantile** of $[X(t) \mid X(\theta_{n-1})]$ for $t \to \theta_n$,

(N3) The **ceiling of the median** of $[X(t) \mid X(\theta_{n-1})]$ for $t \to \theta_n$, i.e.

$$N_1 := \left\lceil \lim_{t \to \theta_1} x_0 \exp\left(H_0^{\xi_2}(\theta_0, t)\right) \right\rceil, \; N_n := \left\lceil \lim_{t \to \theta_n} N_{n-1} p_{n-1} \exp\left(H_{n-1}^{\xi_2}(\theta_{n-1}, t)\right) \right\rceil, k \geq 2. \tag{10}$$

The log-likelihood function of the process $X(t)$

Let us consider a **discrete sampling** of the process $X_k(t)$, for $k \in \mathbb{N}_0$, based on d sample paths at times $t_{i,j}$, $j \in \{1, \ldots, n_i\}$, $i \in \{1, \ldots, d\}$. We denote by

$$\mathbf{X}^{(k)} = \left(\left(\mathbf{X}_1^{(k)} \right)^{\mathsf{T}}, \ldots, \left(\mathbf{X}_d^{(k)} \right)^{\mathsf{T}} \right)^{\mathsf{T}}$$

the vector of all observed data, where $\mathbf{X}_i^{(k)} = \left(X_k(t_{i,1}^{(k)}), \ldots, X_k(t_{i,n_i}^{(k)}) \right)^{\mathsf{T}}$, $i \in \{1, \ldots, d\}$. For simplicity, let us consider $t_{i,1}^{(k)} = \theta_k$ for any $i = 1, \ldots, d$. Denoting with $\boldsymbol{\beta}^{(k)}$ the vector containing the parameters involved in the definition of $[X_k(t) \mid X_k(\theta_k) = y]$ and with $\boldsymbol{\eta}^{(k)} = (p_k, N_k)$, we have that the joint PDF of $\mathbf{X}^{(k)}$ has the following expression

$$f_{\mathbf{X}}^{(k)}\left(\boldsymbol{x}^{(k)}\right) = \prod_{i=1}^{d} \binom{N_k}{x_{i,1}^{(k)}} p_k^{x_{i,1}^{(k)}} (1 - p_k)^{N_k - x_{i,1}^{(k)}} \prod_{j=1}^{n_i - 1} \frac{\exp\left(- \frac{\left[\log \frac{x_{i,j+1}^{(k)}}{x_{i,j}^{(k)}} - m_k^{i,j+1,j} \right]^2}{2\sigma^2 \Delta_i^{j+1,j}} \right)}{x_{i,j}^{(k)} \sigma \sqrt{2\pi \Delta_i^{j+1,j}}},$$

where $\boldsymbol{x} \in \mathbb{R}_+^{n+d}$, $n = H_k^{\boldsymbol{\xi}_2}(t_{i,j}, t_{i,j+1})$ and $\Delta_i^{j+1,j} := t_{i,j+1} - t_{i,j}$ for any $j \in \{1, \ldots, n_i\}$, $i \in \{1, \ldots, d\}$ and $k \in \mathbb{N}_0$.

By considering the following change of variables

$$V_{0,i}^{(k)} = X_{i,1}^{(k)}, \qquad i = 1, \ldots, d$$

$$V_{i,j}^{(k)} = (t_{i,j+1} - t_{i,j})^{-1/2} \log\left(\frac{X_{i,j+1}^{(k)}}{X_{i,j}^{(k)}}\right), \qquad j = 1, \ldots, n_i - 1; \quad i = 1, \ldots, d,$$

we are able to obtain an explicit expression for the log-likelihood function of $\boldsymbol{V}^{(k)}$

$$L_{\boldsymbol{V}}(\boldsymbol{\beta}^{(k)}, \boldsymbol{\eta}^{(k)}) = \sum_{i=1}^{d} \log\binom{N_k}{v_{0,i}^{(k)}} + \log p_k \sum_{i=1}^{d} v_{0,i}^{(k)} + \log(1-p_k) \sum_{i=1}^{d} (N_k - v_{0,i}^{(k)})$$
$$- \frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{Z_1 + \phi - 2\Gamma}{2\sigma^2},$$

where

$$Z_1 = \sum_{i=1}^{d} \sum_{j=1}^{n_i - 1} (v_{i,j}^{(k)})^2, \qquad \phi = \sum_{i=1}^{d} \sum_{j=1}^{n_i - 1} \frac{(m_k^{i,j+1,j})^2}{\Delta_i^{j+1,j}}, \qquad \Gamma = \sum_{i=1}^{d} \sum_{j=1}^{n_i - 1} \frac{v_{i,j}^{(k)} m_k^{i,j+1,j}}{(\Delta_i^{j+1,j})^{1/2}}.$$

Parameters Estimation Algorithm

Initialize the observed data
Phase 1: Parameter estimation
- For each time interval:
  Estimate the parameters (η, β, σ) using the maximum likelihood method via Nelder Mead algorithm
Phase 2: Determination of $\hat{N}_n$
- If N_n is known:
  Use the given N_n value
- Otherwise, if N_n is unknown:
  Use the deterministic rule to estimate N_n by considering the estimates of the parameters obtained in Phase 1
Phase 3: Estimation of the binomial probability $\hat{p}_B$
- Compute m_n as the sample mean of the observed data
- Compute $\hat{p}_B = \frac{m_n}{\hat{N}_n}$
Return the estimated values $(\hat{\eta}, \hat{\beta}, \hat{\sigma}, \hat{N}_n, \hat{p}_B)$

Estimation strategy (simulation study)

Goal

For each simulated dataset, we estimate the model parameters via **maximum likelihood.**

Method 1

Nelder–Mead (N.M.)

- numerical optimization method
- implemented in `R` (`optim`)

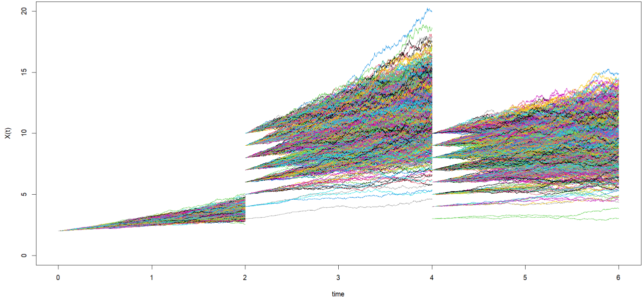
Method 2

Genetic Algorithm (G.A.)

- evolutionary optimization method
- implemented in `R` (`GA`)

Simulated sample paths

We assume that the binomial size is known *a priori*, fixed, and constant across cycles, i.e. $N_n = N = 10$ for all $n \in \mathbb{N}$. the others parameters are set as follows $p_B \in \{0.75, 0.8\}$, $\eta \in \{e^{-1}, e^{-3/2}\}$, $\beta \in \{0.4, 0.5\}$, $\sigma \in \{0.01, 0.05\}$. For the choice of the boundaries, we consider: (B1) $p_C = 0.5$, (B2) $k = 1.05$, (B3) $\alpha_1 = -0.5$, $\alpha_2 = 0.5$.



Effect of the number of observations on the estimation accuracy

Obs.	Method	Case (B1)			Case (B2)			Case (B4)		
		$RAE(\eta)$	$RAE(\beta)$	$RAE(\sigma)$	$RAE(\eta)$	$RAE(\beta)$	$RAE(\sigma)$	$RAE(\eta)$	$RAE(\beta)$	$RAE(\sigma)$
100	GA	0.14020	0.15835	<u>0.04584</u>	0.40161	0.19215	0.05574	0.11486	0.12805	<u>0.01590</u>
100	N.M.	0.00938	<u>0.00252</u>	0.29084	0.00373	0.00105	0.29669	0.00057	0.02306	0.29706
50	GA	0.13416	0.17235	0.02398	0.38821	0.18316	<u>0.02762</u>	0.10503	0.11892	0.05835
50	N.M.	<u>0.00076</u>	0.00276	0.29978	<u>0.00220</u>	<u>0.00060</u>	0.30046	<u>0.00001</u>	<u>0.00021</u>	0.30048
25	GA	0.49721	0.49721	0.09754	0.41091	0.19057	0.16369	0.12916	0.10116	0.13754
25	N.M.	0.02861	0.18790	0.32248	0.00438	0.00133	0.30711	0.00004	0.00004	0.30664

RAE for $X_0 = 2$, $\eta = e^{-1}$, $\beta = 0.5$, $\sigma = 0.01$, $p_C = 0.5$, $p_B = 0.8$, $N = 10$ (known).
For each row, the smallest RAE is underlined.

Effect of observations on p_B

Observations	Case (B1)	Case (B2)	Case (B4)
	$RAE(p_B)$	$RAE(p_B)$	$RAE(p_B)$
100	<u>0.00111</u>	<u>0.00351</u>	<u>0.00058</u>
50	0.00387	0.00663	0.00093
25	0.67128	0.01367	0.00102

Table: RAE of p_B using N.M. for $X_0 = 2, \eta = e^{-1}, \beta = 0.5, \sigma = 0.01, p_C = 0.5, p_B = 0.8,\ N = 10$ (known), varying observations.

- Increasing the number of observations generally improves the estimation of p_B.
- Case (B4) appears particularly robust across all scenarios.

Known N: RAE of parameters (Table 2)

p_B	η	β	σ	case (B1)			case (B2)			case (B4)		
				$RAE(\eta)$	$RAE(\beta)$	$RAE(\sigma)$	$RAE(\eta)$	$RAE(\beta)$	$RAE(\sigma)$	$RAE(\eta)$	$RAE(\beta)$	$RAE(\sigma)$
0.8	e^{-1}	0.5	0.01	0.00497	0.01812	0.29524	0.00080	0.00033	0.29597	0.00089	0.00095	0.29637
			0.05	0.00183	0.00789	0.29597	0.00132	0.00112	0.29594	0.00476	0.00297	0.29656
		0.4	0.01	0.00073	0.00078	0.29594	0.00047	0.00013	0.29600	0.00028	0.00046	0.29647
			0.05	0.00831	0.00575	0.29575	0.00127	0.00189	0.29601	0.00582	0.00195	0.29634
	$e^{-3/2}$	0.5	0.01	0.00022	0.00005	0.29605	0.00099	0.00033	0.29600	0.00018	0.00015	0.29633
			0.05	0.00267	0.00056	0.29614	0.00525	0.00286	0.29594	0.00497	0.00045	0.29634
		0.4	0.01	0.00044	0.00012	0.29631	0.00106	0.00022	0.29596	0.00026	0.00014	0.29626
			0.05	0.00448	0.00027	0.29613	0.00262	0.00253	0.29587	0.00658	0.00025	0.29645
0.75	e^{-1}	0.5	0.01	0.00009	0.00061	0.29451	0.00069	0.00020	0.29600	0.00096	0.00106	0.29652
			0.05	0.00390	0.01101	0.29573	0.00098	0.00088	0.29596	0.00503	0.00144	0.29650
		0.4	0.01	0.04689	0.01599	0.28232	0.00083	0.00034	0.29585	0.00040	0.00041	0.29647
			0.05	0.00716	0.01444	0.29587	0.00365	0.00111	0.29601	0.00560	0.00042	0.29627
	$e^{-3/2}$	0.5	0.01	0.00028	0.00014	0.29629	0.00068	0.00020	0.29592	0.00037	0.00024	0.29667
			0.05	0.00254	0.00024	0.29616	0.00291	0.00259	0.29589	0.00501	0.00053	0.29627
		0.4	0.01	0.00033	0.00001	0.29634	0.00141	0.00038	0.29609	0.00025	0.00012	0.29655
			0.05	0.00448	0.00053	0.29615	0.00263	0.00255	0.29594	0.00629	0.00002	0.29644

Table: RAE of parameters via N.M. for $N_k = 10$ known (simulation study). Lowest RAE underlined per row.

Known N: RAE for p_B (Table 3)

p_B	η	β	σ	case (B1) $RAE(p_B)$	case (B2) $RAE(p_B)$	case (B4) $RAE(p_B)$
0.8	e^{-1}	0.5	0.01	0.00221	0.00303	<u>0.00010</u>
			0.05	<u>0.00088</u>	0.00407	0.00107
		0.4	0.01	0.00158	0.00411	<u>0.00054</u>
			0.05	0.00190	0.00434	<u>0.00134</u>
	$e^{-3/2}$	0.5	0.01	<u>0.00097</u>	0.00443	0.00111
			0.05	<u>0.00062</u>	0.00481	0.00124
		0.4	0.01	0.00090	0.00446	<u>0.00046</u>
			0.05	0.00095	0.00437	<u>0.00010</u>
0.75	e^{-1}	0.5	0.01	0.00303	0.00481	<u>0.00012</u>
			0.05	0.00194	0.00486	<u>0.00181</u>
		0.4	0.01	0.00231	0.00444	<u>0.00005</u>
			0.05	0.00214	0.00472	<u>0.00140</u>
	$e^{-3/2}$	0.5	0.01	<u>0.00044</u>	0.00499	0.00203
			0.05	<u>0.00074</u>	0.00512	0.00212
		0.4	0.01	0.00182	0.00468	<u>0.00044</u>
			0.05	0.00090	0.00496	<u>0.00010</u>

Table: RAE for p_B via N.M. for $N_k = 10$ known (simulation study).

Simulation study: take-home messages

- Estimation quality is good across (B1), (B2), (B4): boundary choice does not drastically change performance.
- (B2) can produce worse estimates for η when catastrophes occur too early (limited information about saturation).
- Increasing observations usually improves accuracy, especially for p_B.
- N.M. is the best trade-off here: fast and accurate in most settings.

Application to wolf popultaion

Analysis of the wolf population on Isle Royale (1969–1986), with catastrophic events occurring at $\theta_1 = 1977$ and $\theta_2 = 1982$.

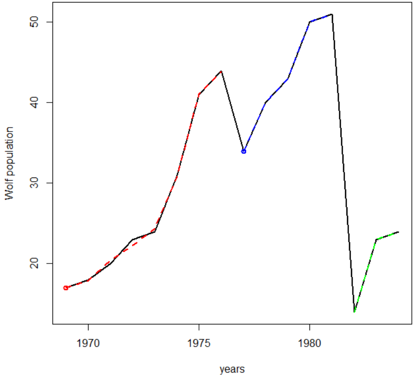


Figure: Observed wolf population on Isle Royale from 1969 to 1986. The two catastrophic drops correspond to the change points $\theta_1 = 1977$ and $\theta_2 = 1982$.

Polynomial Degree Selection Procedure

The maximum degree for $Q_\beta(t)$ is fixed at 6

↓

Define model class $\mathfrak{M}_j$ for $j = 1, \ldots, 6$

↓

Select model with highest posterior probability

The methodology adopted is based on the approach described in[4].

Bayesian procedure

Given the dataset (t, x^j) from model class $\mathfrak{M}_j$, the posterior probability of model M_j is defined as

$$\mathrm{P}(M_j \mid t, x) = \frac{B_{j,1}(t,x)\pi(M_j)}{\sum_{i=1}^{6} B_{i,1}(t,x)\pi(M_i)}, \qquad j = 1, \dots, 6$$

where $\pi(M_j) = \frac{1}{6}$ is the prior probability related to the selection of the model M_j (assumed to be uniform),

$$B_{i,1}(t,x) = \frac{2}{\pi}(i+1)^{(i-1)/2} \int_0^{\pi/2} \frac{\sin^{i-1}(\varphi)\left(n + (i+1)\sin^2(\varphi)\right)^{(n-i)/2}}{\left(n\mathcal{B}_{i,1} + (i+1)\sin^2(\varphi)\right)^{(n-1)/2}} \mathrm{d}\varphi,$$

is the Bayes factor for the comparison between the model M_i and the model M_1, and $\mathcal{B}_{i,1} = \frac{\sum_{k=1}^{n}\left(m_k - \widehat{m}_k^{(i)}\right)^2}{\sum_{k=1}^{n}\left(m_k - \widehat{m}_k^{(1)}\right)^2}$, being $n = \sum_{i=1}^{d}(n_i - 1)$ the total number of data, m_k the sample mean at the k-th time instant and $m_k^{(j)}$ with $j = 1, \dots, 6$ the estimated mean at the k-th time instant obtained by using the model M_j.

Some results

Degree	1969–1976		1977–1981		1982–1984	
	MSE	Posterior Probability	MSE	Posterior Probability	MSE	Posterior Probability
1	3.79916	0.05949	1.02427	0.13344	$\underline{1.80689 \times 10^{-12}}$	—
2	1.99782	0.09243	0.96845	0.06424	—	—
3	0.92089	0.15098	$\underline{3.34008 \times 10^{-8}}$	$\underline{0.80232}$	—	—
4	$\underline{0.11715}$	$\underline{0.49640}$	—	—	—	—
5	0.16912	0.13614	—	—	—	—
6	0.18101	0.06456	—	—	—	—

Table: Mean Squared Error (MSE) and posterior probabilities for polynomial degrees. Best models per period are underlined.

arrival time of the catastrophe	$\widehat{p}_B$	$\widehat{N}$
$\theta_1 = 1977$	$\hat{p}_B^{(1)} = 0.77273$	$\widehat{N}_1 = 22$
$\theta_2 = 1982$	$\hat{p}_B^{(2)} = 0.26923$	$\widehat{N}_2 = 26$

Table: Estimates of the parameters of the binomial distribution.

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