A variance-based approach to study the importance of components and subgroups in a coherent system: the regression importance signature

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Notation

Let $\phi(x_1, \dots, x_n)$ be a coherent system having lifetime $T = \phi(X_1, \dots, X_n)$ and lifetime components $(X_1, \dots, X_n)$. For any $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, we denote with $\boldsymbol{X}_k^* = \{X_{i_1}, \dots, X_{i_k}\}$ and $\boldsymbol{x}_k^* = \{x_{i_1}, \dots, x_{i_k}\} \in \mathbb{R}_+^k$.
Let $\widehat{C}$ be the survival copula of $(X_1, \dots, X_n)$, i.e.,

$$\Pr(X_1 > x_1, \dots, X_n > x_n) = \widehat{C}(\overline{F}_1(x_1), \dots, \overline{F}_n(x_n)), \qquad \boldsymbol{x}_n^* \in \mathbb{R}_+^n. \tag{1}$$

We denote with $\partial_k \widehat{C}$ the partial derivative of $\widehat{C}$ with respect to its k-th argument, and with $\widehat{C}_P$ the survival copula associated with the set P. Taking into account the vector $\mathbf{u} = (u_1, \dots, u_n)$, we write $\mathbf{u}_P := (u_1^P, \dots, u_n^P)$, where $u_i^P = u_i$ if $i \in P$, and $u_i^P = 1$ otherwise. Under these settings, the function $\widehat{C}_{k,P}$ is defined by $\widehat{C}_{k,P}(\mathbf{u}) := \partial_k \widehat{C}(\mathbf{u}_{P \cup \{k\}})$.
Similarly, if C is the copula of $(X_1, \dots, X_n)$, i.e.,

$$\Pr(X_1 \leq x_1, \dots, X_n \leq x_n) = C(F_1(x_1), \dots, F_n(x_n)), \qquad \boldsymbol{x}_n^* \in \mathbb{R}_+^n, \tag{2}$$

then one has $C_{k,P}(\mathbf{u}) := \partial_k C(\mathbf{u}_{P \cup \{k\}})$.

Representation of a coherent system

Let $\phi(x_1, \ldots, x_n)$ be a coherent system having lifetime components $(X_1, \ldots, X_n)$.

- **Minimal path set representation**:

$$T = \phi(X_1, \ldots, X_n) = \max_{j=1,\ldots,r} \min_{i \in P_j} X_i, \tag{3}$$

  where $P_1, \ldots, P_r \subseteq \{1, \ldots, n\}$ are the minimal path sets of the system. A set $P \subseteq \{1, \ldots, n\}$ is a *path set* if the system works whenever all components in P work. It is a *minimal path set* if it does not contain any smaller path set.

- **Minimal cut set representation**:

$$T = \phi(X_1, \ldots, X_n) = \min_{i=1,\ldots,s} \max_{j \in D_i} X_j, \tag{4}$$

  where $D_1, \ldots, D_s \subseteq \{1, \ldots, n\}$ are the minimal cut sets of the system. A set $D \subseteq \{1, \ldots, n\}$ is a *cut set* if the system fails only if all components in D fail. It is a *minimal cut set* if it does not contain any smaller cut set.

Useful stochastic orders

Definition 1

Let X_i be a random variable with CDF F_i and SF $\overline{F}_i$, for $i = 1, 2$. We say that X_1 is smaller than X_2 according to

(i) the **usual stochastic order**, denoted by $X_1 \leq_{st} X_2$, if $\overline{F}_1(t) \leq \overline{F}_2(t)$, for all $t \in \mathbb{R}$;

(ii) the **dispersive order**, denoted by $X_1 \leq_{disp} X_2$, if $F_2^{-1}(\alpha) - F_1^{-1}(\alpha)$ increases in $\alpha \in (0, 1)$, where F_1^{-1} and F_2^{-1} denote the right-continuous inverses of F_1 and F_2, respectively;

(iii) the **convex order**, denoted by $X_1 \leq_{cx} X_2$, if $\mathrm{E}[\phi(X_1)] \leq \mathrm{E}[\phi(X_2)]$ for any convex function ϕ.

Moreover, $=_{st}$ denotes the equality in law.

(ii) and (iii) are variability orders:

- if $X_1 \leq_{disp} X_2$, then $Var(X_1) \leq Var(X_2)$;
- if $X_1 \leq_{cx} X_2$, then $Var(X_1) \leq Var(X_2)$.

Useful notions about copulas

Definition 2

A random vector $(X_1, \dots, X_n)$ is said to be conditionally increasing in sequence (CIS) if and only if $\{X_i \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}\} \leq_{st} \{X_i \mid X_1 = x_1', \dots, X_{i-1} = x_{i-1}'\}$ for any $x_j \leq x_j'$ and $i, j \in \{1, \dots, n\}$ such that $j < i$. Moreover, if $(X_{\pi(1)}, \dots, X_{\pi(n)})$ is CIS for all permutations π, then $(X_1, \dots, X_n)$ is said to be conditionally increasing (CI).

Definition 3

Given two copulas C and C', we say that C is smaller than C' according to the concordance order, denoted by $C \prec C'$, if $C(u, v) \leq C'(u, v)$ for any $u, v \in (0, 1)$.

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Regression curve of a component by using the path sets

Let $(X_1,\ldots,X_n)$ be the lifetime components of a coherent system having absolutely continuous joint PDF and survival copula $\widehat{C}$. Let P be a path set, with $X_P=\min_{j\in P}X_j$. By recalling Lemma 2.1 and Lemma 2.2 in Arriaza et al. [1], for any $k=1,\ldots,n$ and $x\in\mathbb{R}_+$, one has

$$\mathrm{E}[X_P\,|\,X_k=x]=\begin{cases}\displaystyle\int_0^x \widehat{C}_{k,P}\left(\overline{F}_1(y_1),\ldots,\overline{F}_n(y_n)\right)\mathrm{d}t, & \text{if } k\in P,\\ \displaystyle\int_0^{+\infty} \widehat{C}_{k,P}\left(\overline{F}_1(y_1),\ldots,\overline{F}_n(y_n)\right)\mathrm{d}t, & \text{if } k\notin P,\end{cases}\quad x\in\mathbb{R}_+,$$

where $y_k=x$ and $y_j=t$ for $j\neq k$. Moreover, if T is the lifetime of the system and $P_1,\ldots,P_r\subseteq\{1,\ldots,n\}$ are its minimal path sets, then for any $x\in\mathbb{R}_+$ and $k=1,\ldots,n$ the *regression curve of the k-th component by using the minimal path set representation* is defined as

$$\mathbf{m_k(x)}=\mathrm{E}[\mathbf{T}\,|\,\mathbf{X_k}=\mathbf{x}]$$

$$=\sum_{j=1}^{r}\mathrm{E}\left[X_{P_j}|X_k=x\right]-\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\mathrm{E}\left[X_{P_i\cup P_j}|X_k=x\right]+\ldots(-1)^{r+1}\mathrm{E}\left[X_{P_1\cup\ldots\cup P_r}|X_k=x\right],$$

Regression curve of a component by using cut sets

Let $(X_1, \ldots, X_n)$ be the lifetime components of a coherent system having absolutely continuous joint PDF and copula C. Let D be a cut set, with $X_D = \max_{j \in D} X_j$. For any $k = 1, \ldots, n$ and $x \in \mathbb{R}_+$, it holds

$$\mathrm{E}[X_D \mid X_k = x] = \begin{cases} x + \displaystyle\int_x^{+\infty} [1 - C_{k,D}(F_1(y_1), \ldots, F_n(y_n))] \, \mathrm{d}t., & \text{if } k \in D, \\ \displaystyle\int_0^{+\infty} [1 - C_{k,D}(F_1(y_1), \ldots, F_n(y_n))] \, \mathrm{d}t, & \text{if } k \notin D. \end{cases} \quad x \in \mathbb{R}_+,$$

where $y_k = x$ and $y_j = t$ for $j \neq k$. Hence, if T is the system lifetime and $D_1, \ldots, D_s$ are its minimal cut sets, then, for any $k = 1, \ldots, n$ and $x \in \mathbb{R}_+$, the *regression curve by using the minimal cut set representation* is given by

$$m_k(x) = \sum_{i=1}^{s} \mathrm{E}[X_{D_i} \mid X_k = x] - \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} \mathrm{E}[X_{D_i \cup D_j} \mid X_k = x] + \ldots + (-1)^{s+1} \mathrm{E}[X_{D_1 \cup \ldots \cup D_s} \mid X_k = x].$$

Regression importance index

Definition 4 (Definition 3.1 in Arriaza et al. [1])

Given an n-component coherent system, the **regression importance index of the k-th component** is defined as

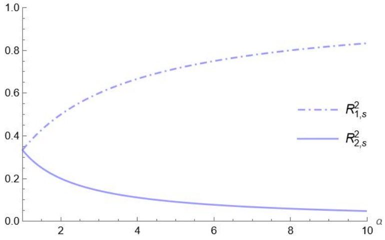
$$0 \leq R_k^2 = \frac{\mathrm{Var}(m_k(X_k))}{\mathrm{Var}(T)} \leq 1, \qquad k = 1, \dots, n. \tag{5}$$

- Intuitively, the greater the proportion of the variance in the system's lifetime T that can be predicted from the k-th component, the more relevant its role in the system.
- The importance of a component is determined not only by its reliability, but also by the role it plays within the system.

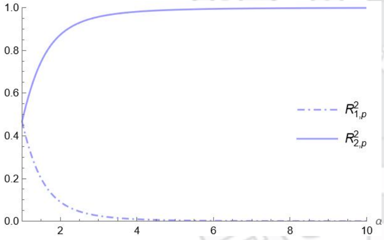
An example to better understand the index

Let us consider a 2-component system having exponential distributed lifetime components (X_1, X_2) with parameters $\lambda_1 = \alpha\lambda_2$, for $\alpha \in [1, +\infty)$.

$$\Rightarrow X_1 \leq_{st} X_2$$



(a)



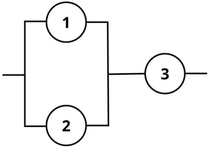
(b)

Figure: The importance indices of (X_1, X_2) organized (a) in series and (b) in parallel, with respect to $\alpha > 1$.

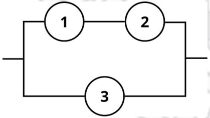
Other examples

Table: Importance indices for coherent systems of size 3 with exponential IID lifetime components. For each system, when components have different indices, the largest is underlined.

	System	$\phi(X_1,\ldots,X_n)$	$(R_1^2,\ R_2^2,\ R_3^2)$
(i)	Series	$\min\{X_1,X_2,X_3\}$	(0.200, 0.200, 0.200)
(ii)		$\max\{\min\{X_1,X_3\},\min\{X_2,X_3\}\}$	(0.067, 0.067, <u>0.567</u>)
(iii)	2-out-of-3	$\max\{\min\{X_1,X_2\},\min\{X_1,X_3\},\min\{X_2,X_3\}\}$	(0.246, 0.246, 0.246)
(iv)		$\max\{\min\{X_1,X_2\},X_3\}$	(0.024, 0.024, <u>0.873</u>)
(v)	Parallel	$\max\{X_1,X_2,X_3\}$	(0.302, 0.302, 0.302)



(a) System in case (ii).



(b) System in case (iv).

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Regression curves for set of components

Lemma 5

Let $X_1, \dots, X_n$ be the lifetime components of a coherent system with a joint absolutely continuous distribution and survival copula $\widehat{C}$. Let P be a path set and $X_P = \min_{j \in P} X_j$. Let $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and $\boldsymbol{x}_k^* \in \mathbb{R}_+^k$ be such that $f_{\boldsymbol{X}_k^*}(\boldsymbol{x}_k^*) > 0$. Under suitable conditions on $\widehat{C}_{i_1,\dots,i_k}$, one has

$$\Pr(X_P > t \,|\, \boldsymbol{X}_k^* = \boldsymbol{x}_k^*) = \frac{\widehat{C}_{i_1,\dots,i_k,P}(\overline{F}_1(y_1), \dots, \overline{F}_n(y_n))}{\widehat{C}_{i_1,\dots,i_k}(\overline{F}_{i_1}(x_{i_1}), \dots, \overline{F}_{i_k}(x_{i_k}))} \mathbf{1}_{\{t < \mathrm{m}_A\}},$$

where $A = \{i_1, \dots, i_k\} \cap P$, $\mathrm{m}_A = \min_{j \in A} x_j$, $y_i = x_i$ for $i \in \{i_1, \dots, i_k\}$, and $y_j = t$ for $j \in P \setminus A$. Moreover,

$$\mathrm{E}(X_P \,|\, \boldsymbol{X}_k^* = \boldsymbol{x}_k^*) = \int_0^{\mathrm{m}_A} \frac{\widehat{C}_{i_1,\dots,i_k,P}(\overline{F}_1(y_1), \dots, \overline{F}_n(y_n))}{\widehat{C}_{i_1,\dots,i_k}(\overline{F}_{i_1}(x_{i_1}), \dots, \overline{F}_{i_k}(x_{i_k}))} \mathrm{d}t.$$

Lemma 6

Let $(X_1, \dots, X_n)$ be the lifetime components of a coherent system having absolutely continuous joint PDF and survival copula $\widehat{C}$. Let $P \subseteq \{1, \dots, n\}$ be such that $X_P = \min_{j \in P} X_j$. Let $\{i_1, \dots, i_k\} \subseteq \{1, \dots . n\} \backslash P$ and $\boldsymbol{x}_k^* \in \mathbb{R}_+^k$ be such that $f_{\boldsymbol{X}_k^*}(\boldsymbol{x}_k^*) > 0$. Under suitable conditions on $\widehat{C}_{i_1,\dots,i_k}$, one has

$$\Pr(X_P > t \,|\, \boldsymbol{X}_k^* = \boldsymbol{x}_k^*) = \frac{\widehat{C}_{i_1,\dots,i_k,P}(\overline{F}_1(y_1), \dots, \overline{F}_n(y_n))}{\widehat{C}_{i_1,\dots,i_k}(\overline{F}_{i_1}(x_{i_1}), \dots, \overline{F}_{i_k}(x_{i_k}))},$$

where $y_i = x_i$ for $i \in \{i_1, \dots, i_k\}$ and $y_j = t$ for $j \in P \setminus \{i_1, \dots, i_k\}$. Moreover,

$$\mathrm{E}(X_P \,|\, \boldsymbol{X}_k^* = \boldsymbol{x}_k^*) = \int_0^{+\infty} \frac{\widehat{C}_{i_1,\dots,i_k,P}(\overline{F}_1(y_1), \dots, \overline{F}_n(y_n))}{\widehat{C}_{i_1,\dots,i_k}(\overline{F}_{i_1}(x_{i_1}), \dots, \overline{F}_{i_k}(x_{i_k}))} \mathrm{d}t.$$

Regression importance index for set of components

Let $X_1, \ldots, X_n$ be the lifetime components of a coherent system having lifetime T. By recalling Lemmas 5 and 6, for any $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and $\boldsymbol{x}_k^* = x_1 \ldots, x_k \in \mathbb{R}_+^k$, the *regression curve* of $\boldsymbol{X}_k^* = (X_{i_1}, \ldots, X_{i_k})$ is given by

$$\mathbf{m_{i_1,\ldots,i_k}}(\mathbf{x_k^*}) = \mathrm{E}[\mathbf{T} \mid \mathbf{X_k^*} = \mathbf{x_k^*}]$$

$$= \sum_{i=1}^{r} \mathrm{E}[X_{P_i} \mid \boldsymbol{X}_k^* = \boldsymbol{x}_k^*] - \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \mathrm{E}[X_{P_i \cup P_j} \mid \boldsymbol{X}_k^* = \boldsymbol{x}_k^*] + \ldots + (-1)^{r+1} \mathrm{E}[X_{P_1 \cup \ldots \cup P_r} \mid \boldsymbol{X}_k^* = \boldsymbol{x}_k^*],$$

where $P_1, \ldots, P_r$ are the path/cut sets of the system.

Definition 7

Let $X_1, \ldots, X_n$ be the lifetime components of a coherent system having lifetime T. For any $\{i_1, \ldots i_k\} \subseteq \{1, \ldots, n\}$, the **regression importance index of the i_1-th, ..., i_k-th components** is defined as

$$0 \leq R^2_{i_1,\ldots,i_k} = \frac{\mathrm{Var}\left(m_{i_1,\ldots,i_k}(X_{i_1}, \ldots, X_{i_k})\right)}{\mathrm{Var}(T)} \leq 1. \tag{6}$$

Results on modules

Let $X_1, \ldots, X_n$ be the lifetime components of a coherent system having lifetime $T = \phi(X_1, \ldots, X_n)$. Let $M = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ be a **module** for the system with components $\boldsymbol{x}_M = (x_{i_1}, \ldots, x_{i_k})$ and lifetime $T_M = \chi_M(\boldsymbol{X}_M)$, i.e.,

$$\phi(x_1, \ldots, x_n) = \phi(\boldsymbol{x}) = \Psi(\chi_M(\boldsymbol{x}_M), \boldsymbol{x}_{\overline{M}}),$$

where $\overline{M} = \{1, \ldots, n\} \backslash M$.

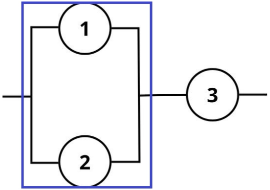
Proposition 1

If $X_1, \ldots, X_n$ are independent, then for any $\boldsymbol{x}_k^* \in \mathbb{R}_+^k$ one has

$$\mathrm{E}\left[T \mid \boldsymbol{X}_M = \boldsymbol{x}_k^*\right] = \mathrm{E}\left[T^\dagger \mid T_M = \chi_M(\boldsymbol{x}_k^*)\right], \tag{7}$$

where $T^\dagger = \Psi(\chi_M(\boldsymbol{X}_M), \boldsymbol{X}_{\overline{M}})$ is the lifetime of the coherent system having the same structure of T, but the components of the module M are replaced by a single component with lifetime T_M.

An example...



Let X_1, X_2, X_3 be independent lifetime components of a system having lifetime $T = \max\{\min\{X_1, X_3\}, \min\{X_2, X_3\}\}$.
Since $M = \{1, 2\}$ is a parallel system and a module, then

$$T = \min\{T_M, X_3\},$$

where $T_M = \max\{X_1, X_2\}$ is the lifetime of the module M.

From Lemma 5, it follows

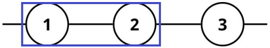
$$m_{1,2}(x_1, x_2) = \int_0^{\max\{x_1,x_2\}} \overline{F}_3(t)\,\mathrm{d}t \qquad x_1, x_2 \in \mathbb{R}_+.$$

If we consider M as a new unique component, then it holds

$$m_M(x) = \mathrm{E}[X_{\{M,3\}} \mid T_M = x] = \int_0^x \overline{F}_3(t)\mathrm{d}t, \qquad x \in \mathbb{R}_+.$$

$$\Rightarrow m_{1,2}(X_1, X_2) = m_M(T_M).$$

...and a counterexample



Consider a 3-component series system having exponential lifetime components X_i, with parameters $\lambda_i > 0$, for $i = 1, 2, 3$.

Assume that X_1 is independent from X_2 and X_3, while there is a dependence structure between X_2 and X_3, governed by the FGM copula

$$\widehat{C}(u, v) = uv[1 + \theta(1 - u)(1 - v)], \qquad \theta \in [-1, 1], \;\; u, v \in [0, 1]. \tag{8}$$

Clearly, $M = \{1, 2\}$ is a module forming a 2-series system, i.e., $T_M = \min\{X_1, X_2\}$. For $x_1, x_2, x \in \mathbb{R}_+$, one has

$$m_{1,2}(x_1, x_2) = \frac{1 - e^{-\lambda_3 \min\{x_1, x_2\}}}{\lambda_3} + \theta\,(1 - 2e^{-\lambda_2 x_2})\,\frac{1 - 2\,e^{-\lambda_3 \min\{x_1,x_2\}} + e^{-2\lambda_3 \min\{x_1,x_2\}}}{2\,\lambda_3},$$

$$m_M(x) = \frac{1 - e^{-\lambda_3 x}}{\lambda_3} + \theta\left(1 - \frac{\lambda_1 + 2\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_2 x}\right)\frac{1 - 2e^{-\lambda_3 x} + e^{-2\lambda_3 x}}{2\,\lambda_3}.$$

$$\Rightarrow m_{1,2}(X_1, X_2) \neq m_M(T_M).$$

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Comparisons between importance indices

Let $X_1, \dots, X_n$ be independent lifetime components of a coherent system having lifetime T.

- If $T = \min\{X_1, \dots, X_n\}$ and $X_1 \leq_{disp} \dots \leq_{disp} X_n$, then

$$R_1^2 \geq \dots \geq R_n^2,$$

  i.e., the weakest component has the highest coefficient.

- If $T = \max\{X_1, \dots, X_n\}$ and $X_1 \leq_{disp} \dots \leq_{disp} X_n$, then

$$R_1^2 \leq \dots \leq R_n^2,$$

  i.e., the strongest component has the highest coefficient.

More in general:

Let $X_1, \ldots, X_n$ be independent lifetime components of a coherent system having lifetime $T = \phi(X_1, \ldots, X_n)$.

Proposition 2

From Eqs. (5) and (6), let R_i^2 be the importance index of the i-th component, $R_{i,j}^2$ the importance index of (X_i, X_j), and so on. For any $i, j, k = 0, 1, \ldots, n$, it holds

$$R_i^2 \leq R_{i,j}^2 \leq R_{i,j,k}^2 \leq \ldots \leq R_{1,\ldots,n}^2 = 1. \tag{9}$$

Lemma 8 (Generalization of Theorem 4.12 in Arriaza et al. [1])

Let $C^{(k)}$ denote the copula of the vector (T, X_k), for $k = 1, \ldots, n$. If $(X_1, \ldots, X_n)$ is CI and $C^{(i)} \prec C^{(j)}$, then $m_i(X_i) \leq_{cx} m_j(X_j)$ for any $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. It follows that

$$R_i^2 \leq R_j^2.$$

Lemma 9

Let $T = \min\{X_1, \ldots, X_n\}$ *be the lifetime of a series system having independent lifetime components* X_i*, for* $i = 1, \ldots, n$*. Let* M_i *be a module of the system and* $X_{M_i} = \min_{j \in M_i} X_j$*, for* $i = 1, 2$*. If* $X_{M_1} \leq_{st} X_{M_2}$*, then* $m_{M_2}(X_{M_2}) \leq_{cx} m_{M_1}(X_{M_1})$ *and thus*

$$R^2_{M_2} \leq R^2_{M_1}.$$

Moreover, if $X_{M_1} \leq_{disp} X_{M_2}$*, then* $m_{M_1}(X_{M_1}) \geq_{disp} m_{M_2}(X_{M_2})$*. Hence,* $R^2_{M_1} \geq R^2_{M_2}$*.*

Lemma 10

Let $T = \max\{X_1, \ldots, X_n\}$ *be the lifetime of a parallel system having independent lifetime components* X_i*, for* $i = 1, \ldots, n$*. Let* M_i *be a module of the system and* $X_{M_i} = \min_{j \in M_i} X_j$*, for* $i = 1, 2$*. If* $X_{M_1} \geq_{st} X_{M_2}$*, then* $m_{M_2}(X_{M_2}) \leq_{cx} m_{M_1}(X_{M_1})$*, and thus*

$$R^2_{M_1} \geq R^2_{M_2}.$$

Moreover, if $X_{M_1} \geq_{disp} X_{M_2}$*, then* $m_{M_1}(X_{M_1}) \geq_{disp} m_{M_2}(X_{M_2})$*. Hence,* $R^2_{M_1} \geq R^2_{M_2}$*.*

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The regression importance signature

Definition 11

Let $X_1, \dots, X_n$ be the lifetime components of a coherent system having lifetime $T = \phi(X_1, \dots, X_n)$. The **regression importance signature** of the system is a vector

$$(R^2_{\{1\}}, R^2_{\{2\}}, \dots, R^2_{\{n-1\}}),$$

where

$$R^2_{\{k\}} = \max_{i_1,\dots,i_{k-1}} R^2_{i_1,\dots,i_{k-1}} \quad \text{for} \quad i_1, \dots, i_{k-1} \in \{1, \dots n\} \quad \text{s.t.} \quad i_1 < \dots < i_{k-1}.$$

is the maximum among the importance indices of all the possible subgroups of k components, for $k \in \{1, \dots, n\}$.

Since $\mathrm{Var}(m_{1,\dots,n}(X_1, \dots, X_n)) = \mathrm{Var}(T)$, one has $R^2_{\{n\}} = R^2_{1,\dots,n} = 1$, so it is omitted.

Examples

Table: Importance signature for coherent system of size 3 in Table 1 with exponential IID lifetime components, where $R^2_{\{1\}} = \max_i R^2_i$ and $R^2_{\{2\}} = \max_{j>i} R^2_{i,j}$.

	$R^2_{\{1\}}$	$R^2_{\{2\}}$
(i)	$R^2_i = 0.200, \forall i$	$R^2_{i,j} = 0.500, \forall i,j$
(ii)	$R^2_3 = 0.567$	$R^2_{1,3} \equiv R^2_{2,3} = 0.750$
(iii)	$R^2_i = 0.246, \forall i$	$R^2_{i,j} = 0.537, \forall i,j$
(iv)	$R^2_3 = 0.873$	$R^2_{1,3} \equiv R^2_{2,3} = 0.912$
(v)	$R^2_i = 0.302, \forall i$	$R^2_{i,j} = 0.633, \forall i,j$

For instance, case (ii) corresponds to the coherent system having lifetime $T = \min\{\max\{X_1, X_2\}, X_3\}$, whose regression importance signature is

$$(R^2_{\{1\}}, R^2_{\{2\}}) = (R^2_3, R^2_{1,3} \equiv R^2_{2,3}) = (0.567, 0.750).$$

Importance signature for well-known coherent systems

Let $R^2_{i_1,\ldots,i_k}$ be defined as in Eq. (6) for any $\{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\}$.

Proposition 3

If $T = \min\{X_1,\ldots,X_n\}$ *is the lifetime of a series system having independent lifetime components and* $X_1 \leq_{st} \ldots \leq_{st} X_n$*, then*

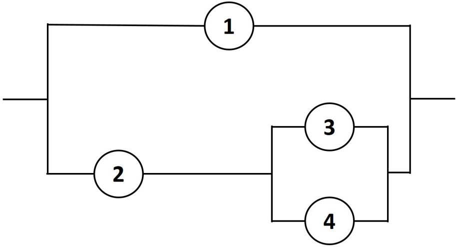
$$\left(R^2_{\{1\}},\, R^2_{\{2\}},\, \ldots,\, R^2_{\{n-1\}} \right) = \left(R^2_{1},\, R^2_{1,2}, \ldots, R^2_{1,\ldots,n-1} \right).$$

Proposition 4

If $T = \max\{X_1,\ldots,X_n\}$ *is the lifetime of a parallel system having independent lifetime components and* $X_1 \geq_{st} \ldots \geq_{st} X_n$*, then*

$$\left(R^2_{\{1\}},\, R^2_{\{2\}},\, \ldots,\, R^2_{\{n-1\}} \right) = \left(R^2_{1},\, R^2_{1,2}, \ldots, R^2_{1,\ldots,n-1} \right).$$

Case study of a ship control system



Let $T = \max\{X_1, \min\{X_2, X_3\}, \min\{X_2, X_4\}\}$ (see Arriaza et al. 1).

- X_1: lifetime of the manual control valves;
- X_2: lifetime of the electric motor;
- X_3: lifetime of the bridge control panel;
- X_4: lifetime the machine control panel.

There is a slight dependence among the components due to the fact that they share the same marine environment, described by the FGM copula:

$$C(u_1, u_2, u_3, u_4) = u_1\, u_2\, u_3\, u_4\, [1 + \theta\,(1-u_1)\,(1-u_2)\,(1-u_3)\,(1-u_4)],$$

where $\theta \in [-1, 1]$ and $u_i \in [0, 1]$, for $i = 1, 2, 3, 4$.

Applying the Monte Carlo method, we generate four exponential random samples of size $n = 10^6$ with different parameters in order to simulate the lifetime components X_i, for any $i = 1, 2, 3, 4$.

(i) If $\lambda_1 = 1/60 < \lambda_2 = 1/50 < \lambda_3 = \lambda_4 = 1/45$, then $X_1 \geq_{st} X_2 \geq_{st} X_3 =_{st} X_4$ and the regression importance signature is

$$(R^2_{\{1\}}, R^2_{\{2\}}, R^2_{\{3\}}) = (R^2_1 , R^2_{1,2} , R^2_{1,2,3} \approx R^2_{1,2,4}). \quad (10)$$

(ii) If $\lambda_1 = 1/15 > \lambda_2 = 1/45 > \lambda_3 = \lambda_4 = 1/60$, then $X_1 \leq_{st} X_2 \leq_{st} X_3 =_{st} X_4$ and the regression importance signature is

$$(R^2_{\{1\}}, R^2_{\{2\}}, R^2_{\{3\}}) = (R^2_2 , R^2_{2,3} \approx R^2_{2,4} , R^2_{2,3,4}). \quad (11)$$

The importance indices are computed by means of RStudio software and the implemented algorithm has time-complexity $O(n^3)$.

Table: Importance indices for each subgroup of k components, with $1 \leq k \leq 3$, for the coherent system studied in our case study, considering parameters (i) in (a) and (ii) in (b). The indices forming the regression importance signatures given respectively in Eqs. (10) and (11) are underlined.

	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
R_1^2	$\underline{0.894}$	$\underline{0.892}$	$\underline{0.889}$
R_2^2	0.029	0.030	0.031
R_3^2	0.006	0.005	0.005
R_4^2	0.006	0.005	0.005
$R_{1,2}^2$	$\underline{0.942}$	$\underline{0.941}$	$\underline{0.941}$
$R_{1,3}^2$	0.902	0.904	0.905
$R_{1,4}^2$	0.902	0.903	0.905
$R_{2,3}^2$	0.049	0.049	0.049
$R_{2,4}^2$	0.049	0.049	0.049
$R_{3,4}^2$	0.013	0.013	0.014
$R_{1,2,3}^2$	$\underline{0.968}$	$\underline{0.966}$	$\underline{0.964}$
$R_{1,2,4}^2$	$\underline{0.968}$	$\underline{0.966}$	$\underline{0.964}$
$R_{1,3,4}^2$	0.913	0.912	0.912
$R_{2,3,4}^2$	0.072	0.071	0.074

(a)

	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
R_1^2	0.077	0.075	0.073
R_2^2	$\underline{0.538}$	$\underline{0.541}$	$\underline{0.543}$
R_3^2	0.042	0.041	0.041
R_4^2	0.042	0.041	0.041
$R_{1,2}^2$	0.645	0.647	0.649
$R_{1,3}^2$	0.119	0.121	0.122
$R_{1,4}^2$	0.120	0.121	0.122
$R_{2,3}^2$	$\underline{0.684}$	$\underline{0.685}$	$\underline{0.686}$
$R_{2,4}^2$	$\underline{0.683}$	$\underline{0.684}$	$\underline{0.685}$
$R_{3,4}^2$	0.106	0.107	0.108
$R_{1,2,3}^2$	0.791	0.792	0.793
$R_{1,2,4}^2$	0.791	0.792	0.793
$R_{1,3,4}^2$	0.187	0.189	0.196
$R_{2,3,4}^2$	$\underline{0.886}$	$\underline{0.885}$	$\underline{0.884}$

(b)

Index

Concluding Remarks

Key points of our work:

- Extension of the regression importance index to subgroup of not necessarily independent components, with a particular attention to system modules;
- Introduction of a regression importance signature, in order to identify, for each k, the subgroup of k components that deserves the most attention;
- Study of conditions for comparing individual components and subgroups.

What about the future?

- Exploring possible connection with other well-known signatures;
- Construction of a dynamic version of the regression importance index for evaluating the variation over time in the importance of each component;
- Computation of the regression importance signature for networks having particular structure.

Main references

[1] Arriaza A., Navarro J., Sordo M.A., and Suárez-Llorens, A. (2023) A variance-based importance index for systems with dependent components. *Fuzzy Set Syst.* **467**, 108482. DOI: https://doi.org/10.1016/j.fss.2023.02.003.

[2] Di Crescenzo A., Pisano G., Suárez-Llorens, A. (2025) Analysis of systems with dependent components through a variance-based index and regression importance signature. *Currently under revision.*

[3] Navarro J. (2022) *Introduction to System Reliability Theory*. Springer, Cham. DOI: https://doi.org/10.1007/978-3-030-86953-3.

[4] Nelsen R.B. (2006) *An Introduction to Copulas* (2nd ed.). Springer, New York. DOI: https://doi.org/10.1007/0-387-28678-0.

[5] Shaked, M. and Shanthikumar, J.G. (2007) *Stochastic Orders*. Springer Series in Statistics. Springer, New York. DOI: https://doi.org/10.1007/978-0-387-34675-5.

Thank for your attention!