ABP estimates and first exit time in cylindrical domains

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Stochastic Models for Biomathematics and Applications

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Overview

First exit time from a cylindrical domain $\Omega \subset \mathbb{R}^n$

Estimates of moments

A fully nonlinear non-uniformly elliptic operator

Stochastic processes and first exit time

Let us consider a stocastic process X_t^x, $t \geq 0$, in domain Ω of $\mathbb{R}^n$ starting from $x \in \Omega$, namely $X_0^x = x$.
The *first exit time* or *first passage time* FPT is the random variable

$$\tau_\Omega^x := \inf\{t \geq 0 : X_t^x \notin \Omega\}.$$

Letting $\mathscr{L}$ be the (infinitesimal) generator of the process, the mean first exit time $u(x) = \mathbb{E}(\tau_\Omega^x)$ is the solution of the Dirichlet problem

$$\begin{cases} \mathscr{L}u = -1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

If the process is governed by the SDE

$$\Delta X_t^x = b(X_t^x)dt + \boldsymbol{\sigma}(X_t^x)dW_t$$

with drift b and diffusion matrix $\boldsymbol{\sigma}$, and $A(x) = \boldsymbol{\sigma}(x)\boldsymbol{\sigma}^T(x)$, then

$$\begin{aligned} \mathscr{L}u &= \tfrac{1}{2}\operatorname{Tr}\left(A(x)\nabla^2 u\right) + b(x)\cdot\nabla u \\ &= \tfrac{1}{2}\, a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} \end{aligned}$$

First exit time in cylindrical domains

Applications in cylindrical domains include, among the others, the so-called *narrow escape problem* for a Brownian particle (neurotransmitter) to reach a target (receptor) located on the basis of a cylinder of large radius. Other applications in chemical reactions, fluid dynamics and heat transfer in a pipe can be considered.

■ In a recent paper (Di Crescenzo-Spina-V., Differ. Integral Equ. '24) we proved a existence and uniqueness results for more general, fully nonlinear equations, with continuous boundary conditions, in more general domains, namely

$$\begin{cases} \mathscr{F}u = f(x) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{DP}$$

where

$$\mathscr{F}u = F(x, u, \nabla u, \nabla^2 u),$$

and Ω is a domain, possibly unbounded, satisfying condition

$$\textit{finite inner radius} \; + \; \textit{uniform exterior cone condition} \tag{GEC}$$

which we also call *cylindrical domains*.

Cylindrical domains: unbounded

• *Inner radius*: $R_{in} := \sup\{R > 0 : B_R(x) \subset \Omega \text{ for some } x \in \Omega\}$

• *Exterior cone condition*: for all $x \in \partial\Omega$ there exist a finite circular cone Γ_x such that $\overline{\Gamma}_x \subset \mathscr{C}\Omega$, uniform if Γ_x are rigid transformation of a fixed cone.

Lemma. *If Ω satisfies condition* (**GEC**)*, then there is a covering of Ω with ball of radius $R \le R_0$ such that*

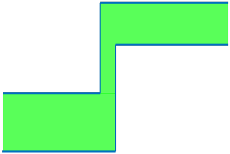
$$|B_R \backslash \Omega| \ge \sigma |B_R| \qquad (\tilde{\mathbf{G}})$$

for some $\sigma \in (0, 1)$ and $R_0 \in \mathbb{R}_+$.

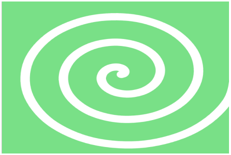
• The parameter R_0, which can be taken as the diameter of Ω for bounded domains, is mostly influencing on the mean first exit time.

Cylindrical domains: unbounded

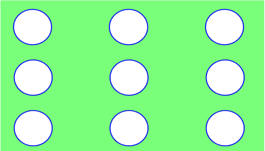
infinite cylinders and slabs



connected unions



complement of thick Archimedean spirals



complement of balls

Elliptic equations

Fully nonlinear equations $\mathscr{F}u = f(x)$ under consideration are defined by means of fully nonlinear operators

$$F : (x,t,\xi,X) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to F(x,t,\xi,X) \in \mathbb{R},$$

where $\mathcal{S}^n$ is the space of $n \times n$ real symmetric matrices, assuming the structure condition

$$\begin{aligned}
&F \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n) \\
&F(x,t,\xi,X) - F(x,t,\eta,Y) \le \mathcal{M}^+_{\lambda,\Lambda}(X-Y) + b(x)|\xi - \eta| \\
&F(x,t,\xi,X) - F(x,s,\xi,X) \le 0 \ \ \textit{if} \ \ s < t \\
&F(x,0,0,0) = 0
\end{aligned} \qquad (\mathbf{SC})$$

where $b(x)$ is non-negative, continuous and bounded.

Here $\mathcal{M}^+_{\lambda,\Lambda}$ is the Pucci maximal operator with constants $0 < \lambda \le \Lambda$:

$$\mathcal{M}^+_{\lambda,\Lambda}(X) := \Lambda \sum_{\mathrm{eig}(X)>0} \mathrm{eig}(X) + \lambda \sum_{\mathrm{eig}(X)<0} \mathrm{eig}(X) = \sup_{A \in \mathscr{S}^n_{\lambda,\Lambda}} \mathrm{Tr}(AX),$$

and $(\mathbf{SC})$ implies uniform ellipticity with ellipticity constants λ and Λ.

Uniform ellipticity

Let $X, Y \in \mathcal{S}^n$, we say that $Y \leq X$ if $X - Y$ is semidefinite positive

- *Uniform ellipticity* means that, if $Y \leq X$, then

$$\lambda \mathrm{Tr}(X - Y) \leq F(x,t,\xi,X) - F(x,t,\xi,Y) \leq \Lambda \mathrm{Tr}(X - Y).$$

It is implied by condition (**SC**), and In the linear case $\mathscr{F} = \mathscr{L}$ it reduces to the usual condition:

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

But note that $\mathcal{M}^+_{\lambda,\Lambda}$ itself is uniformly elliptic, as a particular case of Bellman operators.

Note also that uniform ellipticity implies *(degenerate) ellipticity*:

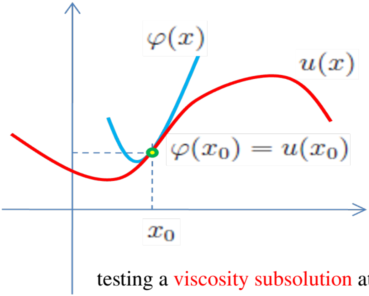
$$Y \leq X \;\Rightarrow\; F(x,t,\xi,Y) \leq F(x,t,\xi,X),$$

namely F non-decreasing with respect to the partial ordering of $\mathcal{S}^n$.

Viscosity solutions (Crandall-Lions)

- A function $u \in \text{usc}(\Omega)$ ($\text{lsc}(\Omega)$) is a viscosity subsoln (supsoln) of the elliptic equation $\mathscr{F}u = f$, for short $\mathscr{F}u \geq f$ ($\mathscr{F}u \leq f$), if for any $x_0 \in \Omega$ and $\varphi \in C^2(B_r(x_0))$ touching u from above (below) at x_0, it is

$$F(x, \varphi(x), \nabla\varphi(x), \nabla^2\varphi(x)) \geq f(x) \quad (\leq f(x)).$$



testing a viscosity subsolution at x_0

- Viscosity solutions are both viscosity sub and supersolutions.
- Classical solutions $u \in C^2(\Omega)$ are viscosity solutions. Conversely, viscosity solutions $u \in C^2(\Omega)$ are classical solutions.

Existence

Theorem (DCSV, '24). *Let Ω be a domain satisfying condition* $(\mathbf{GEC})$ *with parameter R_0, and F be satisfying condition* $(\mathbf{SC})$. *Suppose b, f continuous in Ω, and*

$$\sup_{x\in\Omega}|b(x)| \leq b_0, \quad \|f\|_{R_0} := \sup_{x\in\mathbb{R}^n}\|f\|_{L^n(\Omega\cap B_{R_0}(x))} < \infty.$$

Let $\varphi \in C^0(\partial\Omega)\cap L^\infty(\partial\Omega)$. There exists a viscosity solution $u \in C^0(\overline{\Omega})$ of (DP). Moreover u is bounded, and

$$\sup_{\Omega}|u| \leq \sup_{\partial\Omega}|\varphi| + CR_0\|f\|_{R_0}.$$

where C is a positive constant depending on the dimension n, the structure constants λ, Λ, b_0R_0, and σ.

Supposing in addition $\varphi \in C^{\tilde{\alpha}}(\partial\Omega)$ with $0 < \tilde{\alpha} < 1$, there exists $\alpha \in (0, \tilde{\alpha})$ such that $u \in C^\alpha(\overline{\Omega})$, and:

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C\left(\|\varphi\|_{C^{\tilde{\alpha}}(\partial\Omega)} + R_0\|f\|_{R_0}\right).$$

Arguments of proof

- The proof existence result in unbounded domains is made by approximation on larger and larger bounded domains, by using an existence result of Crandall-Kocan-Lions-Swiech (EJDE '99) in bounded domains and then applying Ascoli-Arzelà theorem.

- The fundamental tool is the Alexandroff-Bakeman-Pucci (ABP) estimate, which provides a uniform bound for the sequence of approximating solutions.

- Equicontinuity is provided by a uniform Hölder estimate which is based on Harnack inequality for non-negative viscosity solutions, due to Caffarelli (Annals '89).

- Uniform convergence is enough, in viscosity setting, in order that the limit function is a solution.

- Boundary condition is obtained via barrier functions based on the uniform exterior cone condition.

ABP estimate (improved)

Theorem (Cabré, CPAM '94; Capuzzo Dolcetta-Leoni-V., CPDE '05). *Let* Ω *be a domain satisfying condition* $(\tilde{\mathbf{G}})$ *with parameter* R_0*, and* F *be continuous such that*

$$\mathcal{M}^{-}_{\lambda,\Lambda}(X) - b_0|\xi| \leq F(x,t,\xi,X) \leq \mathcal{M}^{+}_{\lambda,\Lambda}(X) + b_0|\xi|$$

with $b_0 > 0$*. If* $u \in C^0(\overline{\Omega})$ *is a viscosity solotion of the equation*

$$F(x,u,\nabla u,\nabla^2 u) = f(x) \ \ in\ \Omega,$$

then

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + CR_0 \|f\|_{R_0},$$

where

$$\|f\|_{R_0} := \sup_{x\in\mathbb{R}^n} \|f\|_{L^n(\Omega\cap B_{R_0}(x))},$$

and C *is a positive constant depending on the dimension* n*, the structure constants* λ, Λ*, the geometric constant* σ*, and the mixed product* $b_0 R_0$*.*

If f *is bounded then the above inequality can be written in the form*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + CR_0^2 \|f\|_{L^\infty(\Omega)},$$

Estimate of the mean first exit time

The mean first exit time $\tau_1(x) = \mathbb{E}(\tau_\Omega^x)$ is the solution of the Dirichlet problem

$$\begin{cases} \frac{1}{2}\, a_{ij}(x)\, \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x)\, \frac{\partial u}{\partial x_i} = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

which is covered by the existence theorem, corresponding to:

$$F(x,t,\xi,X) = \frac{1}{2}\,\mathrm{Tr}(A(x)X) + b(x)\cdot\xi, \;\; f = -1, \;\; \varphi = 0,$$

and so τ_1 is a continuous function in $\overline{\Omega}$.
Noting that $f = -1$ is negative and bounded, the ABP estimate yields the maximum principle, so coherently $\tau_1 \geq 0$ in Ω, and the estimate

$$\sup_{\Omega} \tau_1 \leq C R_0^2,$$

according to the parabolic scaling: for $\gamma > 0$

$$\sup_{x\in\gamma\Omega} \mathbb{E}(\tau_{\gamma\Omega}^x) \leq \gamma^2 C R_0^2.$$

Explicit solutions in a slab (large basis and small height)

In a slab $S = \{x \in \mathbb{R}^n : -d < x_n < d\}$, we have

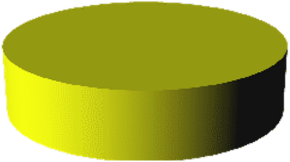
$$\sup_{\Omega} \tau_1 \leq C d^2$$

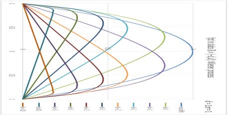
We computed solutions for Brownian motion with constant drift $b_n = b$:

no drift ($b = 0$) $\qquad \tau_1(x) = \frac{d^2 - x_n^2}{\sigma^2}$

drift ($b \neq 0$) $\qquad \tau_1(x) = \frac{d - x_n}{b} - \frac{2d}{b} \frac{e^{\frac{2b}{\sigma^2}(d - x_n)} - 1}{e^{\frac{4b}{\sigma^2} d} - 1}.$

Plotting it, we see that any drift reduces the mean FPT apart on a layer close to the boundary hyperplane opposite to drift.





Explicit solutions in a pipe (large height and small basis)

In a cylinder $C = \{x \in \mathbb{R}^n : |x'| < r\}$, we have

$$\sup_{\Omega} \tau_1 \leq Cr^2.$$

For Brownian motion with constant axial-symmetric drift $b(x) = b\,\frac{x'}{r}$:

no drift $(b = 0)$ $$\tau_1(x) = \frac{r^2 - |x'|^2}{(n-1)\sigma^2}$$

drift $(b \neq 0)$ $$\tau_1(x) = \frac{r - |x'|}{b} - \frac{\sigma^2}{2b^2} \int_{|x'|}^{r} \frac{1 - e^{\frac{2b}{\sigma^2}\rho}}{\rho}\, d\rho$$

where a detailed analysis shows that τ_1 decreases if $b > 0$ and increases if $b > 0$, as expected.

Estimates of higher moments

- Note also that, based on the continuity of F, by assumptions, the ellipticity of a_{ij} and the boundedness of b_i also yield the uniqueness of the continuous solution τ_1 (Crandall, Ishii, Lions, '92).

Higher moments of the first exit time

$$\tau_k(x) = \int_0^\infty t^k \, dp_\Omega^x(t)$$

satisfy, for $k = 2, 3, \ldots$, the recursive equation

$$\begin{cases} \frac{1}{2} a_{ij}(x) \frac{\partial^2 \tau_k}{\partial x_i \partial x_j} + b_i(x) \frac{\partial \tau_k}{\partial x_i} = -k\tau_{k-1} & \text{in } \Omega \\ \tau_k = 0 & \text{on } \partial\Omega. \end{cases}$$

which is still covered by the existence theorem, and by induction, the moments τ_k are continuous in $\overline{\Omega}$, and from ABP estimate we get

$$\sup_\Omega \tau_k \le k! \, C^k \, R_0^{2k},$$

which also contains the estimate of τ_1 for $k = 1$.

Partial trace operators

Let $\lambda_i(X)$, $i = 1, \ldots, n$, be the eigenvalues of $X \in \mathcal{S}^n$ arranged in non-decreasing order. If $(i_1, \ldots, i_k)$ is a choice of $k < n$ eigenvalues among the λ_i's, the following operator is called partial trace operator:

$$\lambda_{i_1}(X) + \cdots + \lambda_{i_k}(X),$$

which appears in geometric problems of mean partial curvature.
Among such operators, we focus on

$$M(X) = \lambda_1(X) + \lambda_n(X),$$

which has also an interpretation in two-players zero-sum stochastic differential games. A detailed study of such operator is made in Ferrari-V. (Adv. Nonlinear Stud., '20), where the following min-max representation is established:

$$M(X) = \sup_{|\xi|=1} \inf_{|\eta|=1} \mathrm{Tr}(X_{\xi,\eta}) = \inf_{|\eta|=1} \sup_{|\xi|=1} \mathrm{Tr}(X_{\xi,\eta}),$$

with $X_{\xi,\eta}$ the matrix of the quadratic form associated to X when restricted to the plane spanned by $\xi, \eta \in \mathbb{R}^n$.

ABP estimate for non-uniformly elliptic equations

For $n = 2$ we have $M(\nabla^2 u) = \Delta u$, the Laplace operator.

- But for $n \geq 3$ it is neither linear nor uniformly elliptic. In fact, if

$$X_1 = e_1 \otimes e_1 - e_3 \otimes e_3, \quad X_2 = e_1 \otimes e_1 - e_2 \otimes e_2,$$

then

$$M(X_1 + X_2) = 1, \quad M(X_1) + M(X_2) = 0;$$

and, if $X_3 = e_1 \otimes e_1 - e_2 \otimes e_2 - e_3 \otimes e_3$, then

$$X_3 \leq X_1, \quad M(X_1) - M(X_3) = 0, \quad \mathrm{Tr}(X_1 - X_3) = 1.$$

- Nonetheless,

$$\mathcal{M}^-_{\frac{1}{n},\frac{n+1}{n}}(X) \leq M(X) \leq \mathcal{M}^+_{\frac{1}{n},\frac{n+1}{n}}(X),$$

and this is sufficient to provide the above ABP estimate for viscosity solutions, in a domain Ω satisfying condition $(\tilde{\mathbf{G}})$, of equation

$$M(X) + b_i(x) \cdot \nabla u + c(x)u = f(x)$$

with $\lambda = \frac{1}{n}$ and $\Lambda = \frac{n+1}{n}$.

Further existence

Taking a closer look to partial trace operators, and using viscosity methods, we can prove existence for non-uniformly elliptic equations similar to that of mean first exit time, but driven by M, and so not covered by the previous existence theorem.

Corollary. *Let Ω be a domain satisfying condition* (**GEC**) *with parameter R_0. Suppose b, f continuous in Ω, s.t.*

$$\sup_{x\in\Omega}|b(x)| \leq b_0, \quad \|f\|_{R_0} := \sup_{x\in\mathbb{R}^n}\|f\|_{L^n(\Omega\cap B_{R_0}(x))} < \infty.$$

There exists a unique solution $u \in C^0(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} \frac{1}{2}M(\nabla^2 u) + b(x)\cdot\nabla u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfying the uniform estimate

$$\sup_{\Omega}|u| \leq CR_0\|f\|_{R_0},$$

and in addition the global Hölder estimate $\|u\|_{C^\alpha(\overline{\Omega})} \leq CR_0\|f\|_{R_0}$.