A brief summary of the main aspects of the **long range voter model** and the **p−voter model**

Salvatore dello Russo

Department of Physics
University of Salerno

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Outline

1. Research Group
2. The point of view of Statistical Mechanics
3. The Ordering Problem
4. The **long range** (LR) **voter model** and the **p−voter model**

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The "smal" group

Members: Prof. Federico Corberi, Luca Smaldone, Salvatore dello Russo (2nd year PhD student)

Research Area: Theoretical Physics – Statistical Mechanics

We are all working on the **Department of Physics** of the Univerisity of Salerno.

The Idea of the talk

I will try to give you an **overview** of the reason why we study the models addressed in this four papers:

1. Corberi, F., and Castellano, C., "**Kinetics of the one-dimensional voter model with long-range interactions**", *J. Phys.: Complexity* **5**, 025021 (2024).
2. Corberi, F., and Smaldone, L., "**Ordering kinetics of the two-dimensional voter model with long-range interactions**", *Phys. Rev. E* **109**, 034133 (2024).
3. Corberi, F., **dello Russo, S.**, and Smaldone, L., "**Coarsening and metastability of the long-range voter model in three dimensions**", *Phys. Rev. E* **110**, 024143 (2024).
4. Corberi, F., **dello Russo, S.**, and Smaldone, L., "**Ordering kinetics with long-range interactions: interpolating between voter and Ising models**", *J. Stat. Mech.: Theory Exp.* **2024**, 093206 (2024).

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The Purpose of Statistical Mechanics

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The Purpose of Statistical Mechanics

- The systems under consideration (sample of matter) involve an Avogadro's number of particles, making a particle-by-particle analysis of their mutual interactions impossible in practice.
- To obtain an effective description, **stochastic models** and **statistical methods** are employed to capture the collective behavior of the phenomena under study

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Equilibrium vs Non-Equilibrium

- **Equilibrium statistical mechanics** provides general tools to study the statics of many-particle systems.
- If the interactions between particles are known, and therefore the system Hamiltonian H is specified, there exists a general prescription to describe equilibrium states, summarized by the Boltzmann weight:

$$e^{-\beta H}.$$

- In **non-equilibrium statistical mechanics**, no universal theory exists that applies to all systems.
- As a consequence, each system must be studied on a case-by-case basis, which makes this field particularly **challenging and interesting**.

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Non-equilibrium processes: the **ordering problem**

- We study **non-equilibrium processes** using **stochastic models**.
- Among the big number of non-equilibrium processes, we chose to investigate one of the simplest and **fully non-trivial** ones, which can be considered as prototype: the process related to **ordering problem**.
- The simplest ordering problem involves **Boolean variables**, which can take only two values.

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For example, the **full non-triviality** of the model lies in the fact that it **doesn't equilibrate in finite time**; therefore, the study of its dynamics is the only meaningful approach for such systems.

The Ordering Problem in Statictical Mechanics

- This type of problem is related to physics. If we take a ferromagnet, this is a collection of magnetic moments (commonly called **spins**). These spins interact with each other in a way that favors **alignment** and therefore ($\Rightarrow$) **ordering**.
- However, temperature acts as a **disordering agent**: at high temperature, spins become randomly oriented.

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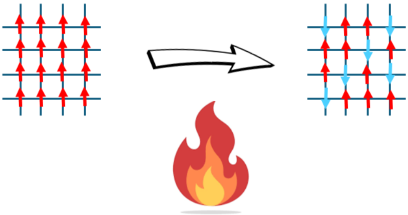
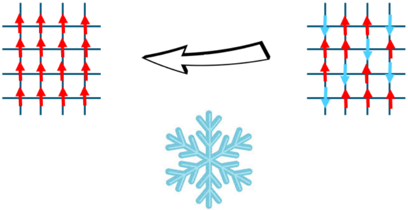
EXAMPLE: everyone knows that if **a magnet** is **heated** enough, it **loses its magnetic properties** and can no longer attract, for example, iron.



The Ordering Problem in Statictical Mechanics

The problem we are interested in is the reverse process, namely
how spins reach order as the temperature decreases.



Axial magnets and the Ising model

- An appropriate model to describe **axial magnets** is the **Ising model**.
- From a dynamical point of view, the model is **not fully solved**.
- An exact analytical solution exists only in:
  - **one dimension (1D)**
  - with **nearest-neighbor interactions**
- The physically relevant cases are **2D and 3D**, since they match the dimensionality of real experimental systems.

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Such problems motivate the search for **simpler and analytically tractable models** that are related to the ordering problem.

Simpler models: the voter model

- Simpler models, originating from **social dynamics**, have been introduced, such as the **voter model**.
- Physicists study these models because statistical mechanicians have tools to tackle stochastic dynamics, which sociologists typically do not.
- The voter model can be seen as a **simplified version of an axial magnet**:
  - no associated Hamiltonian,
  - different dynamical universality class,
  - **solvable in any dimension and for general interactions**.
- This makes it particularly interesting from a **theoretical point of view**, especially in 2D and 3D (dimensions that match the experiments).

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One-dimensional **Long-range Voter Model**

- The model is defined in terms of Boolean (spin) random variables

$$S_i = \pm 1, \qquad i = 1, \ldots, N,$$

placed on a one-dimensional lattice with **periodic boundary conditions**, that can evolve over time.

Update Rule

A spin is chosen uniformly with probability $1/N$. Then another spin at distance r is selected with probability $P(r)$, and the first spin copies the state of the second one, i.e. a randomly chosen spin S_i copies the state of another spin S_j selected at distance r with probability

$$P(r) = \frac{1}{2Z} r^{-\alpha}, \qquad Z = \sum_{r=1}^{N/2} r^{-\alpha} \qquad \alpha > 0.$$

One-dimensional **Long-range Voter Model**

From the previous rule we obtain the **transition probability** that S_i (first spin) flips in a single update is

$$\mathcal{P}_{lrVM}(S_i \to -S_i) = \frac{1}{2N} \sum_{r=1}^{N/2} \left(P(r) \left[\sum_{k=[[i \pm r]]} (1 - S_i S_k) \right] \right), \quad (1)$$

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We are now interested in writing the equation for the correlation function.

The **correlation function** is a very important quantity because it contains information about how strongly two variables of the system are correlated over time.

Correlation function

The **closed equation** for the **correlation function** is

$$\frac{d}{dt}C(r,t) = \frac{d}{dt}\langle S_i(t)S_j(t)\rangle = -2\,C(r,t) + 2\sum_{\ell=1}^{N/2} P(\ell)\left[C([[r-\ell]],t) + C([[r+\ell]],t)\right]$$

where for take into account the **periodic boundary conditions**

$$[[n]] = \begin{cases} |n|, & \text{if } |n| \leq \frac{N}{2}, \\ N-|n|, & \text{if } |n| > \frac{N}{2}. \end{cases}$$

From the knowledge of the correlation function we can obtain important quantities that characterize the system:

- A **correlation length** can be defined as

$$L(t) = \frac{\sum_{r=0}^{N/2} r\, C(r,t)}{\sum_{r=0}^{N/2} C(r,t)}.$$

  It is an indication of the mean distance at which the spins in the system are correlated.

- Another relevant length scale is given by the **density of domain walls** (adjacent opposite spins),

$$\rho(t) = \frac{1}{2}\,[1 - C(r=1,t)].$$

With the same ideas with can deal with the 2D e 3D case of the LR voter model

Main results

The long time behavior of the model depends on the parameter α (exponent in the $P(r)$).

- For large values of α:
  - the behaviour of the model is similar to the **short-range** one,
  - the model reaches **consensus** (absorbing state in which all the spins are in the same state) in the equilibrium state (long-time limit).

- For small values of α:
  - the system assume a **long-range** behaviour,
  - **metastable states** emerge,
  - in the **thermodynamic limit** consensus is not reached in the equilibrium state.

The **transition** between this two kind of behaviour is **sharp**.

Dynamical Universality Class

- Another important result concerns the fact that this model does **not** belong to the same **dynamical universality class** as the long-range Ising model.
- This means that **the exponents governing the dynamics of the two models are different**.
- For instance, if we consider the characteristic length scale $L(t)$, which measures the average distance over which spin states are correlated, in both models one finds an algebraic behaviour

$$L(t) \propto t^{1/z},$$

but the value of the dynamical exponent z is different in the two cases.

p−voter model

Update Rule

A randomly chosen S_i takes the orientation of the partial spin-average

$$\boldsymbol{H_i(\{S_n\}_1^p) = \frac{1}{p}\sum_{n=1}^{p} S_n}$$

of p spins S_n, each chosen at different distances $r_n = |n - i|$ (with probability $P(r_n)$).

For this model the equation fot the correlation in not closed.
Therefore, to study the system we simulate its dynamics using its transition probability.

Transition probability for p−voter model

p is a parameter that characterizes the model

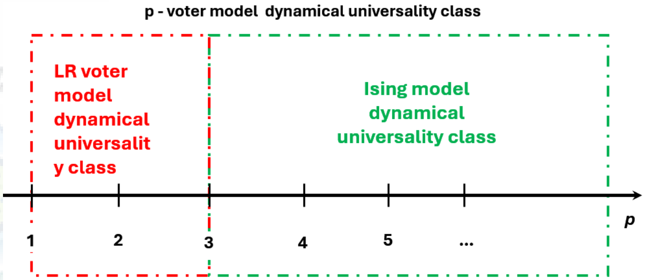
We follow the same operazione of the previous model p times

$$\mathcal{P}_{pVM}(S_i \to -S_i) = [\mathcal{P}_{lrVM}(S_i \to -S_i)]^p$$
$$= \left(\frac{1}{2N} \sum_{r=1}^{N/2} P(r) \sum_{k=[[i \pm r]]} \left[1 - S_i\, \sigma(H_i)\right] \right)^p \qquad (2)$$

where

$$\sigma(H) = \begin{cases} 0, & \text{if } H = 0 \\ \text{sign}(H), & \text{if } H \neq 0. \end{cases}$$

Main results



In general $p \in \mathbb{N}/\{0\}$ therefore, from a dynamical point of view:

- The LR voter model is recovered for $p = 1$;
- The p−voter model for $p = 2$ can be exactly mapped onto the LR voter model;
- For $p \geq 3 \rightarrow$ LR Ising Model universality class.

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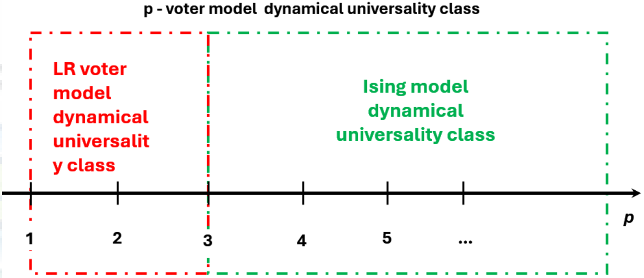

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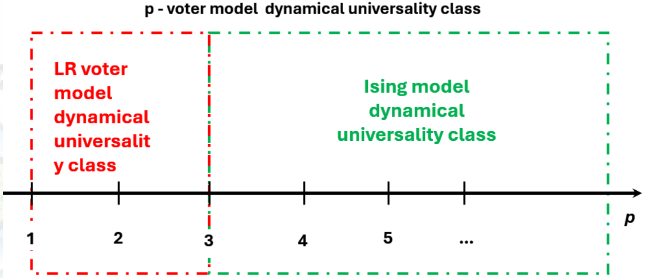

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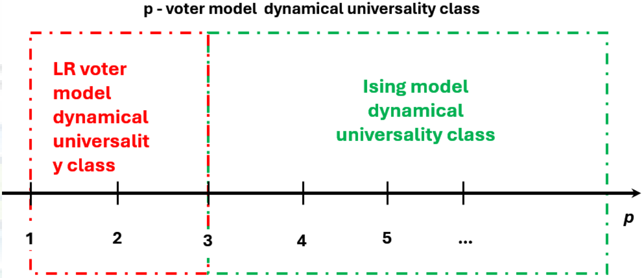

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Conclusion

Due to the fact that the p-voter model belongs to the same universality class as the long-range Ising model, it can be viewed as a simplified version of the long-range Ising model for $p \geq 3$. Although the model is not exactly solvable, it allows for significantly less computationally demanding simulations.

Therefore, **the p-voter model provides a useful tool for investigating the long-range Ising model, which is one of the most important and widely studied models in statistical mechanics**.

END

Thank you all

for your attention!