Cox-Ingersoll-Ross and Bessel diffusion processes: interaction, estimation, applications

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In a certain sense, the squared Bessel process can be considered the result of a phase transition in the Cox-Ingersoll-Ross process. We underline their common and distinct properties. More precisely, we begin by presenting several results that provide upper and lower bounds for the time-asymptotic growth rates of both processes. These bounds exhibit notable similarities between the two models. We explore the approximation of CIR and squared Bessel processes by a sequence of CIR processes. We prove the convergence of this sequence in integral norms, assuming that the corresponding coefficients converge. Additionally, we establish upper bounds on the rate of convergence. It turns out that the CIR and squared Bessel processes are closely related, as the squared Bessel process can be represented as the limit of a sequence of CIR processes. However, as anticipated, the upper bounds involve coefficients that depend on the length of the time interval and tend to infinity as the interval length increases. In this sense, the processes diverge, or, in other words, they move apart. We apply this approximation to the problem of parameter estimation for the squared Bessel process using the maximum likelihood method. To establish the strong consistency of the constructed drift parameter estimator, we approximate the squared Bessel process by a sequence of CIR processes, for which the necessary convergence can be derived from their ergodic properties. Furthermore, we show how to estimate the diffusion coefficient of the process based on the realized quadratic variations. Finally, we investigate both processes using the concept of stochastic instability. From this perspective, the properties of the squared Bessel and CIR processes are fundamentally different. We demonstrate that the squared Bessel process exhibits stochastic instability, whereas the CIR process is ergodic and, in this sense, stochastically stable. The talk is mostly based on the paper Y. Mishura, K. Ralchenko and S. Kushnirenko "Driven by Brownian motion Cox-Ingersoll-Ross and squared Bessel processes: Interaction and phase transition". Physics of Fluids, Vol.37, Iss.1 pp. 1 - 15, - 2025