Lectures on Sasakian Geometry and Topology

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Introduction

Sasakian geometry can be naturally understood as the odd-dimensional counterpart to Kähler geometry. First introduced by Shigeo Sasaki in 1960, the subject has grown from a specialized branch of contact metric geometry into a highly active field of research, deeply intertwined with algebraic geometry, differential topology, and theoretical physics (particularly in the context of string theory and the AdS/CFT correspondence).

A defining feature of a Sasakian manifold is that its metric cone is a Kähler manifold. This elegant link between odd and even dimensions allows one to pull powerful techniques from Kähler geometry down to the Sasakian setting. However, because the fundamental geometric structures are defined on an odd-dimensional space and rely on the existence of a non-vanishing Reeb vector field, the topological implications are fundamentally distinct.

The primary objective of these lecture notes is to provide a concise, self-contained introduction to the geometry and topology of Sasakian manifolds, leading up to a major recent result in the field: the Hard Lefschetz Theorem for Sasakian manifolds. We will carefully develop the machinery required to understand how the local transverse Kähler structure imposes rigid global constraints on the de Rham cohomology of the manifold.

The notes are organized as follows:

- **Chapter 1 (Preliminaries)** reviews the essential prerequisites. We recall the foundational concepts of symplectic, complex, and Kähler geometries in even dimensions, and introduce contact geometry and the Reeb vector field in odd dimensions. We also discuss the symplectization process, which serves as the bridge between the two realms.

- **Chapter 2 (Sasakian Manifolds)** introduces the central objects of this course. We define almost contact metric and K-contact structures, and rigorously formulate the definition of a Sasakian manifold via its metric cone. We then explore equivalent characterizations of Sasakian structures.

- **Chapter 3 (Transverse Geometry)** investigates the geometry orthogonal to the Reeb foliation. We introduce the complex of basic differential forms and basic cohomology. By showing that a Sasakian manifold is transversally Kähler, we establish the transverse Hard Lefschetz Theorem and use the Gysin sequence to link the basic cohomology to the standard de Rham cohomology.

- **Chapter 4 (The Hard Lefschetz Theorem for Sasakian Manifolds)** contains the culmination of these lectures. By exploiting the transverse Hard Lefschetz theorem on basic cohomology alongside the Gysin long exact sequence evaluated at complementary degrees, we construct a Lefschetz-type isomorphism that acts directly on the full de Rham cohomology. Finally, we use this theorem to derive strict parity conditions on the Betti numbers and provide a topological obstruction that separates K-contact manifolds from strictly Sasakian ones.

These notes are designed for graduate students and researchers with a basic background in differential geometry, aiming to quickly reach the frontiers of current research in Sasakian topology.

Chapter 1

Preliminaries

In this chapter, we review the foundational geometric structures that form the prerequisites for Sasakian geometry. We begin by briefly recalling the essentials of symplectic, complex, and Kähler manifolds, before focusing on contact geometry, the Reeb vector field, and the process of symplectization.

1.1 Symplectic Geometry

Symplectic geometry is the natural geometry of classical Hamiltonian mechanics. Unlike Riemannian geometry, it has no local invariants (like curvature), making it a purely topological and global subject.

Definition 1.1.1 (Symplectic Manifold)**.** A **symplectic manifold** is a pair (M, ω), where M is a smooth manifold and $\omega \in \Omega^2(M)$ is a 2-form satisfying two conditions:

1. **Closed:** $d\omega = 0$ (integrability condition);
2. **Non-degenerate:** For each $p \in M$ the map $X_p \mapsto i_{X_p}\omega = \omega(X_p, \cdot)$ is an isomorphism from T_pM to T_p^*M.

Remark 1.1.2. Non-degeneracy inherently forces the dimension of a symplectic manifold to be **even**, say $d = 2n$.[1] An equivalent condition for non-degeneracy is that the top form $\omega^n = \omega \wedge \cdots \wedge \omega$ is nowhere-vanishing. Thus, every symplectic manifold is orientable and comes with a canonical volume form.

Example 1.1.3 (The Standard Model)**.** Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. The standard symplectic form is

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

It is obviously closed ($d\omega_0 = 0$) and non-degenerate.

Example 1.1.4 (Cotangent Bundles)**.** Let Q be any n-dimensional smooth manifold. Its cotangent bundle $M = T^*Q$ carries a canonical symplectic structure. Let $(q_1, \ldots, q_n)$ be

[1] Indeed, if A is the matrix of ω in some basis of T_pM, then $A^T = -A$ due to skew-symmetry, and $\det(A) \neq 0$ due to non-degeneracy. Taking determinants, $\det(A) = \det(A^T) = \det(-A) = (-1)^d \det(A) \implies d$ is even.

coordinates on Q and $(p_1, \ldots, p_n)$ be the fiber coordinates. We define the **tautological 1-form** (or Liouville form) as $\theta = \sum_{i=1}^{n} p_i \, dq_i$. The canonical symplectic form is then the exact 2-form:

$$\omega = d\theta = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Unlike Riemannian geometry, where one can have local curvature (so a sphere is locally distinguishable from a flat plane), symplectic geometry is locally trivial:

Theorem 1.1.5 (Darboux's Theorem)**.** *Let (M^{2n}, ω) be a symplectic manifold. For every point $p \in M$, there exists a local coordinate chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on U:*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

1.2 Complex and Kähler Geometry

To reach Kähler geometry, we must marry our symplectic structure with a rigid local structure: complex geometry.

Definition 1.2.1 (Almost Complex Structure)**.** An **almost complex structure** on a real smooth manifold M is a smooth tensor field $J \in \Gamma(\mathrm{End}(TM))$ such that $J^2 = -I$, where I is the identity endomorphism on the tangent bundle. A manifold with such a structure is called an *almost complex manifold*.

Again, the existence of J requires M to have an even dimension. We can think of J as multiplication by the imaginary unit i on the tangent spaces. An almost complex structure J is said to be **integrable** if it is induced by a genuine complex atlas on M (i.e., coordinate charts to $\mathbb{C}^n$ with holomorphic transition functions). Given a complex chart $(U, z_1, \ldots z_n)$, put $z_h = x_h + iy_h$. One defines J locally by $J\partial_{x_h} = \partial_{y_h}$, $J\partial_{y_h} = -\partial_{x_h}$ and checks that it is actually globally well defined.

More generally, given a real manifold M with an almost complex structure J, there is a criterion that says when the structure is integrable.

Theorem 1.2.2 (Newlander-Nirenberg Theorem)**.** *An almost complex structure J is integrable (making M a **complex manifold**) if and only if the Nijenhuis tensor $N_J \equiv 0$, where:*

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$$

for any vector fields X, Y.

Kähler geometry lies at the beautiful intersection of Riemannian, symplectic, and complex geometry. We now analyze the compatibility condition between the complex structure and the Riemannian metric.

Definition 1.2.3 (Hermitian Metric)**.** A Riemannian metric g on a complex manifold (M, J) is said to be **Hermitian** (or compatible) with respect to J if it is invariant under J, meaning:

$$g(JX, JY) = g(X, Y) \quad \text{for all } X, Y \in TM.$$

Given a Hermitian structure (g, J), we define the **fundamental 2-form** (or Kähler form) ω by:

$$\omega(X, Y) := g(X, JY).$$

It is easy to verify that ω is skew-symmetric:

$$\omega(Y, X) = g(Y, JX) = g(JY, J^2X) = -g(JY, X) = -g(X, JY) = -\omega(X, Y).$$

Furthermore, ω is non-degenerate because g is a non-degenerate metric and J is an isomorphism. Thus, ω is an *almost symplectic form.*

Definition 1.2.4 (Kähler Manifold)**.** A **Kähler manifold** is a Hermitian manifold (M, g, J) such that the fundamental form ω is closed ($d\omega = 0$). In this case, ω is a genuine symplectic form, and M is simultaneously a Riemannian, complex, and symplectic manifold.

The condition that $d\omega = 0$ is quite strong. Equivalently, one can check that a Hermitian manifold (M, g, J) is Kähler if and only if $\nabla J = 0$. In fact, this holds even if the manifold is only almost Hermitian, meaning we do not *a priori* assume the integrability of J, as this is an automatic consequence of J being parallel.

Remark 1.2.5. There are several equivalent definitions of a Kähler manifold. Given any two of the three compatible structures: a Riemannian metric g, an almost complex structure J, and a symplectic form ω, the third is uniquely determined:

- Given g and J, we define $\omega(X, Y) = g(X, JY)$.
- Given ω and J, we define $g(X, Y) = -\omega(X, JY)$.
- Given g and ω, we obtain J via $g(X, JY) = \omega(X, Y)$.

Under the appropriate integrability and compatibility conditions, these structures define a Kähler manifold.

Example 1.2.6 ($\mathbb{C}^n$ as a Kähler Manifold)**.** The complex Euclidean space $\mathbb{C}^n$ is the most fundamental example of a Kähler manifold. It comes equipped with a natural flat metric g, a standard complex structure J, and a standard symplectic form ω.

Let us define the real coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We now specify the three compatible structures.

1. The Riemannian Metric g is the standard Euclidean metric on $\mathbb{R}^{2n}$. In terms of the real coordinates, it is given by:

$$g = \sum_{k=1}^{n} \Big(dx_k \otimes dx_k + dy_k \otimes dy_k \Big) \tag{1.1}$$

2. The almost complex structure $J \colon T\mathbb{C}^n \to T\mathbb{C}^n$ acts as a 90° counterclockwise rotation on the real tangent planes. On the real coordinate basis vector fields, it is defined as:

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}, \quad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k} \tag{1.2}$$

   Applying J twice yields $J^2 = -\mathrm{Id}$, which is the defining property of an almost complex structure.

3. The associated 2-form (the symplectic form) is defined as:

$$\omega(X, Y) = g(X, JY) \tag{1.3}$$

We can compute ω by evaluating it on the real basis vector fields. The resulting 2-form is:

$$\omega = -\sum_{k=1}^{n} dx_k \wedge dy_k \tag{1.4}$$

The triple (g, J, ω) on $\mathbb{C}^n$ fulfills all the requirements of a Kähler manifold:

1. **Hermitian Compatibility:** It immediate to check that $g(JX, JY) = g(X, Y)$.
2. **Integrability of J:** The components of J in $\mathbb{C}^n$ are purely constant. Therefore, its Nijenhuis tensor vanishes trivially, meaning the almost complex structure is integrable (it is enough to evaluate it on the coordinate vector fields).
3. **Closedness of ω:** The symplectic form ω has constant coefficients. Taking the exterior derivative immediately yields $d\omega = 0$.

Because $d\omega = 0$ and J is integrable, $\mathbb{C}^n$ is exactly a Kähler manifold.

Example 1.2.7 (Complex Projective Space)**.** The space $\mathbb{C}P^n$ is the quintessential *compact* Kähler manifold. It carries the so-called **Fubini-Study metric**. The simplest case is in dimension 1, where $\mathbb{C}P^1$ can be thought of as the 2-sphere S^2. It has a Kähler structure with its standard round metric that in local stereographic coordinates is

$$g = \frac{1}{(1 + x^2 + y^2)^2}(dx \otimes dx + dy \otimes dy)$$

and complex structure given by anti-clockwise rotation of 90 degrees ($J(\partial_x) = \partial_y$, $J(\partial_y) = -\partial_x$) Hence

$$\omega = -\frac{1}{(1 + x^2 + y^2)^2}(dx \wedge dy)$$

The reason why one generalizes this as $\mathbb{C}P^n$ instead of S^{2n} is that for topological reasons higher-dimensional spheres do not admit a symplectic structure.

1.3 Contact geometry

We have seen the rich structures present in symplectic geometry, which strictly requires an even-dimensional manifold. A natural question arises: what is the appropriate geometric counterpart in odd dimensions? This leads us to the definition of contact manifolds.

1.3.1 Contact Manifolds

Contact manifolds serve as the natural odd-dimensional counterparts to symplectic manifolds.

Definition 1.3.1 (Contact Form and Distribution)**.** Let M be a manifold of odd dimension $2n+1$. A **contact form** on M is a 1-form η such that:

$$\eta \wedge (d\eta)^n \neq 0 \quad \text{everywhere on } M. \tag{1.5}$$

The **contact distribution** is the hyperplane field $H \subset TM$ defined point-wise as

$$H_p = \ker(\eta_p) \subset T_pM. \tag{1.6}$$

A pair (M, η) is called a **contact manifold**.

Remark 1.3.2. Technically, the most general definition of a **contact structure** is simply a *maximally non-integrable* hyperplane distribution H, independent of any specific 1-form. If H is co-orientable (the quotient line bundle TM/H is trivial), then one can show that H is globally the kernel of a 1-form η. It is not unique: any other 1-form $\eta' = f\eta$ for a nowhere-zero function f defines the same contact structure. **In Sasaki geometry, we fix a specific contact form** η. This is necessary to define the Reeb vector field and the associated metric.

Remark 1.3.3 (Existence of a global contact 1-form)**.** To understand why the triviality of the quotient bundle TM/H gives us a global 1-form, we can construct it directly. Since H has codimension 1, the quotient TM/H is a real line bundle over M. We have a natural projection map $\pi : TM \to TM/H$ whose kernel is exactly H. If TM/H is trivial, there exists a global trivialization map (a bundle isomorphism) $\Phi : TM/H \xrightarrow{\sim} M \times \mathbb{R}$. We can construct a global contact form η by simply composing the projection with this trivialization:

$$\eta = \Phi \circ \pi : TM \to M \times \mathbb{R}.$$

A smooth bundle map from TM to the trivial line bundle $M \times \mathbb{R}$ is precisely the definition of a differential 1-form. By construction, $\ker(\eta) = \ker(\pi) = H$.

1.3.2 The Geometric Meaning: Maximal Non-Integrability

The condition $\eta \wedge (d\eta)^n \neq 0$ says that H is as far as possible from being integrable. More precisely, the 2-form $d\eta$ restricts to a non-degenerate skew-symmetric form on each fiber H_p, so that $d\eta|_H$ is a symplectic form on the $2n$-dimensional vector space H_p.

Theorem 1.3.4 (Frobenius theorem, codimension one case)**.** *Let $H \subset TM$ be a smooth hyperplane distribution. The following are equivalent:*

1. *H is integrable, that is, through every point of M there passes a local submanifold whose tangent spaces coincide with H.*
2. *H is involutive, meaning that for all vector fields $X, Y \in \Gamma(H)$ one has $[X, Y] \in \Gamma(H)$.*
3. *Locally, if $H = \ker \alpha$ for a non-vanishing 1-form α, then $\alpha \wedge d\alpha = 0$.*

Why this matters for contact geometry: Assume that $H = \ker \eta$ and let $X, Y \in \Gamma(H)$. Since $\eta(X) = \eta(Y) = 0$, the exterior derivative gives

$$d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = -\eta([X, Y]).$$

Hence, if H were integrable, then $[X, Y] \in \Gamma(H)$ and therefore $d\eta(X, Y) = 0$ for all $X, Y \in \Gamma(H)$. In other words, integrability of H forces

$$d\eta|_H = 0.$$

But in the contact case, $d\eta|_H$ is non-degenerate at every point, so it cannot vanish. Thus a contact distribution is never integrable.

In the integrable case ($\eta \wedge d\eta = 0$), the manifold is neatly sliced into a deck of $(d-1)$-dimensional "leaves". If you walk along paths strictly tangent to H, you will remain forever trapped on the specific leaf you started on. Instead, in the non-integrable case ($\eta \wedge d\eta \neq 0$), moving along paths tangent to H allows you to "drift" vertically out of the initial plane.

The extreme opposite of the Frobenius condition is when $\eta \wedge (d\eta)^n \neq 0$ maximally; this defines a Contact Structure. In a contact structure, you can reach any point in the manifold by walking strictly along paths tangent to the distribution. This is the sense in which a contact distribution is maximally non-integrable.

1.3.3 Hypersurfaces in Symplectic Manifolds

Now that we understand what a contact manifold is, we can ask how such structures naturally arise. A canonical way to construct a contact manifold is by looking at an odd-dimensional hypersurface M^{2n+1} embedded in a $(2n+2)$-dimensional symplectic manifold (N^{2n+2}, ω).

Because M has odd dimension, the restriction $\omega|_M$ is degenerate: at each point $p \in M$, $\ker(\omega|_M)$ is a 1-dimensional subspace of T_pM (the characteristic foliation). To induce a well-behaved, non-degenerate contact structure on M, we require a vector field on N that generates a scaling of the symplectic structure.

Definition 1.3.5 (Euler Vector Field)**.** A vector field $\mathcal{E}$ on a symplectic manifold (N, ω) is called an **Euler vector field** if it satisfies

$$\mathcal{L}_{\mathcal{E}}\omega = 2\omega, \tag{1.7}$$

that is, the symplectic form is homogeneous of degree 2 with respect to $\mathcal{E}$.

Remark 1.3.6. In general symplectic topology, an Euler (or Liouville) vector field is often defined by $\mathcal{L}_{\mathcal{E}}\omega = \omega$. Here, the factor of 2 is standard in contact and Sasakian geometry because $\mathcal{E}$ acts as the radial Euler field $r\frac{\partial}{\partial r}$ on a cone $C(M) = M \times \mathbb{R}_{>0}$. As we will see later, the symplectic form scales quadratically in r. Furthermore, via Cartan's magic formula and the non-degeneracy of ω, the existence of an Euler vector field is equivalent to the symplectic form being exact.

We can now formally state and prove how a transverse hypersurface inherits a contact structure from an ambient symplectic manifold equipped with an Euler vector field.

Proposition 1.3.7 (Induced Contact Form)**.** *Let (N^{2n+2}, ω) be a symplectic manifold equipped with an Euler vector field $\mathcal{E}$. Suppose $M^{2n+1} \subset N$ is a hypersurface that is* ***transverse*** *to $\mathcal{E}$, meaning that at each point $p \in M$,*

$$T_pN = \langle \mathcal{E}_p \rangle \oplus T_pM. \tag{1.8}$$

Then the restriction of the 1-form $i_{\mathcal{E}}\omega$ to M, defined by

$$\eta = (i_{\mathcal{E}}\omega)|_M, \tag{1.9}$$

is a contact form on M.

Proof. Let $\alpha = i_{\mathcal{E}}\omega$ be the 1-form on N, so that $\eta = \alpha|_M$. Because ω is closed ($d\omega = 0$), Cartan's magic formula yields:

$$d\alpha = d(i_{\mathcal{E}}\omega) = \mathcal{L}_{\mathcal{E}}\omega - i_{\mathcal{E}}(d\omega) = 2\omega.$$

Restricting this to M, we obtain $d\eta = 2\omega|_M$.

To show that η is a contact form, we must prove that the $(2n+1)$-form $\eta \wedge (d\eta)^n$ is nowhere zero on M. Let us first evaluate the corresponding form on the ambient manifold N before restriction:

$$\alpha \wedge (d\alpha)^n = (i_{\mathcal{E}}\omega) \wedge (2\omega)^n = 2^n (i_{\mathcal{E}}\omega) \wedge \omega^n.$$

Recall the Leibniz rule for the interior product applied to a wedge power of a 2-form:

$$i_{\mathcal{E}}(\omega^{n+1}) = (n+1)(i_{\mathcal{E}}\omega) \wedge \omega^n.$$

Substituting this back into our equation gives:

$$\alpha \wedge (d\alpha)^n = \frac{2^n}{n+1} i_{\mathcal{E}}(\omega^{n+1}).$$

Because ω is a symplectic form on a $(2n+2)$-dimensional manifold, the top-form ω^{n+1} is a volume form on N, meaning it is nowhere zero.

By hypothesis, M is transverse to $\mathcal{E}$. This means that for any $p \in M$, if we choose a basis $v_1, \ldots, v_{2n+1}$ for T_pM, the collection $\{\mathcal{E}_p, v_1, \ldots, v_{2n+1}\}$ forms a complete basis for T_pN. Evaluating the ambient volume form on this basis yields:

$$\omega^{n+1}(\mathcal{E}_p, v_1, \ldots, v_{2n+1}) \neq 0.$$

By the definition of the interior product, this is exactly:

$$(i_{\mathcal{E}}\omega^{n+1})(v_1, \ldots, v_{2n+1}) \neq 0.$$

This implies that the restriction of the $(2n+1)$-form $i_{\mathcal{E}}(\omega^{n+1})$ to M is non-vanishing. Finally, restricting our earlier identity to M, we get:

$$\eta \wedge (d\eta)^n = (\alpha \wedge (d\alpha)^n)\,\big|_M = \frac{2^n}{n+1} \left(i_{\mathcal{E}}\omega^{n+1}\right)\big|_M \neq 0.$$

Since $\eta \wedge (d\eta)^n$ is a non-vanishing top-form (a volume form) on M, the restriction η is indeed a valid contact form. □

1.4 Canonical Examples of Contact Manifolds

Example 1.4.1 (The Standard Contact Structure on $\mathbb{R}^{2n+1}$)**.** Let $(x_1, y_1, \ldots, x_n, y_n, z)$ be standard Cartesian coordinates on $\mathbb{R}^{2n+1}$. The standard contact form is defined as:

$$\eta_0 = dz - \sum_{i=1}^{n} y_i \, dx_i.$$

Let us compute the contact volume form. The exterior derivative is:

$$d\eta_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

This is essentially the standard symplectic form on $\mathbb{R}^{2n}$. Taking the n-th exterior power gives:

$$(d\eta_0)^n = n! \, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Wedge this with η_0:

$$\eta_0 \wedge (d\eta_0)^n = \Big(dz - \sum y_i \, dx_i\Big) \wedge (n! \, dx_1 \wedge dy_1 \ldots) = n! \, dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \neq 0.$$

The Darboux Theorem for contact manifolds shows that *every* contact manifold locally looks exactly like this.

Example 1.4.2 (The Standard Sphere S^{2n+1})**.** Consider $\mathbb{C}^{n+1}$ with coordinates $z_k = x_k + iy_k$. Following our conventions, the standard symplectic form is $\omega = -\sum_{k=1}^{n+1} dx_k \wedge dy_k$. The radial vector field $\mathcal{E} = \sum_{k=1}^{n+1} \left(x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k} \right)$ is homogeneous of degree 2 ($\mathcal{L}_{\mathcal{E}} \omega = 2\omega$). The unit sphere S^{2n+1} is transverse to $\mathcal{E}$. The induced contact form is $\eta = (i_{\mathcal{E}} \omega)|_{S^{2n+1}}$. Explicitly:

$$\eta = \sum_{k=1}^{n+1} (y_k \, dx_k - x_k \, dy_k).$$

This is the foundational example for Sasaki geometry.

Example 1.4.3 (The Unit Cotangent Bundle)**.** Let (Q, g) be a smooth $(n+1)$-dimensional Riemannian manifold. Its cotangent bundle T^*Q carries a canonical 1-form that has locally the expression $\theta = \sum p_i \, dq_i$ (where q_i are coordinates on Q and p_i the fiber momenta). The 2-form $\omega = d\theta$ makes T^*Q symplectic. The restriction of θ to the unit sphere bundle

$$S^*Q = \{(q, p) \in T^*Q \mid \|p\|_g = 1\}$$

is a contact form and S^*Q is a contact manifold of dimension $2n + 1$.

1.5 The Reeb Vector Field

Because $\dim M = 2n + 1$ is odd, any 2-form, such as $d\eta$, must have a non-trivial kernel (a skew-symmetric matrix in odd dimensions has determinant zero). The contact condition guarantees this kernel is exactly one-dimensional.

Theorem 1.5.1 (Definition and Existence of the Reeb Vector Field)**.** *Let (M^{2n+1}, η) be a contact manifold. There exists a* ***unique*** *vector field $\xi \in \Gamma(TM)$ satisfying:*

$$i_\xi(d\eta) = 0 \quad (i.e.,\ d\eta(\xi, \cdot) = 0), \tag{1.10}$$
$$\eta(\xi) = 1. \tag{1.11}$$

This vector field ξ is called the ***Reeb vector field****.*

Proof. Because $d\eta|_H$ is a non-degenerate symplectic form on the $2n$-dimensional subspace $H = \ker \eta$, the matrix representing $d\eta$ on TM has rank $2n$. Thus,

$$L := \ker(d\eta) = \{V \in TM \mid i_V d\eta = 0\}$$

is exactly a 1-dimensional line subbundle of the tangent bundle.

Suppose L is contained within H. Then for any $V \in L \setminus \{0\}$, we have $\eta(V) = 0$ and $i_V d\eta = 0$. But this would mean

$$i_V(\eta \wedge (d\eta)^n) = \eta(V)(d\eta)^n - \eta \wedge i_V(d\eta)^n = 0 - 0 = 0.$$

Since $\eta \wedge (d\eta)^n$ is a volume form, it cannot be annihilated by a non-zero vector. This is a contradiction.

Therefore, $L \cap H = \{0\}$. Since L is 1-dimensional and H has codimension 1, we obtain $TM = H \oplus L$. Because η does not vanish on L, we can uniquely scale the generator of L so that $\eta(\xi) = 1$. □

The Reeb vector field gives us a canonical splitting of the tangent bundle:

$$TM = H \oplus \mathbb{R}\xi.$$

1.5.1 Fundamental Properties of the Reeb Flow

Let us examine how the flow of ξ interacts with the contact form using Cartan's magic formula.

Proposition 1.5.2. *The Reeb vector field ξ preserves both the contact form and its exterior derivative:*

1. $\mathcal{L}_\xi \eta = 0$.
2. $\mathcal{L}_\xi d\eta = 0$.

Exercise 1.5.3. *Prove Proposition 1.5.2.*

Let φ_t be the be the local flow generated by ξ. Proposition 1.5.2 implies that $\varphi_t^* \eta = \eta$ wherever the flow is defined. Diffeomorphisms that preserve the contact form exactly are called **strict contactomorphisms**. Thus, the Reeb flow acts by strict contactomorphisms.

1.6 Symplectization: Bridging Contact and Symplectic

We noted that a hypersurface transverse to an Euler vector field in a symplectic manifold inherits a contact structure. It turns out we can always run this process in reverse. This reverse process is vital because, in the next chapter, the Sasaki metric will be defined by putting a Kähler metric on this exact space.

Definition 1.6.1 (Symplectization)**.** Let (M^{2n+1}, η) be a contact manifold. The **symplectization** (or symplectic cone) of M is the $2n+2$-dimensional manifold $C(M) = M \times \mathbb{R}_{>0}$. Let r denote the coordinate on $\mathbb{R}_{>0}$. We define a 1-form λ on $C(M)$ by:

$$\lambda = \frac{1}{2} r^2 \eta.$$

(More precisely, $\lambda = \frac{1}{2} r^2 \pi^* \eta$, where $\pi : M \times \mathbb{R}_{>0} \to M$ is the projection).

Proposition 1.6.2. *The 2-form*

$$\omega = d\lambda = \frac{1}{2} r^2 d\eta + r dr \wedge \eta$$

is a symplectic form on $C(M)$. Furthermore, the vector field $\mathcal{E} = r\frac{\partial}{\partial r}$ is an Euler vector field for ω.

Proof. 1. **Closedness is trivial:** since ω is exact it is also closed.
2. **Non-degeneracy:** We compute ω^{n+1}. Since $(d\eta)^{n+1} = 0$ (being a $2n+2$-form on an $2n+1$-dim. manifold) and $dr \wedge dr = 0$, the only surviving terms in the expansion of $(\frac{1}{2} r^2 d\eta + r dr \wedge \eta)^{n+1}$ will be those containing exactly one $r dr \wedge \eta$ and n terms of $\frac{1}{2} r^2 d\eta$. By binomial expansion:

$$\omega^{n+1} = \frac{n+1}{2^n} r^{2n+1}\, dr \wedge \eta \wedge (d\eta)^n.$$

Since $r > 0$ and $\eta \wedge (d\eta)^n$ is a volume form on M, ω^{n+1} is a nowhere-vanishing top form on $M \times \mathbb{R}_{>0}$. Thus ω is non-degenerate.
The proof that $\mathcal{E}$ is an Euler vector field is left as an exercise. □

Exercise 1.6.3. *Complete the proof of Proposition 1.6.2 by showing that the vector field $\mathcal{E} = r\frac{\partial}{\partial r}$ is an Euler vector field for ω.*

Remark 1.6.4. Given a contact manifold (M, η), the cone $C(M) = M \times \mathbb{R}_{>0}$ also admits a symplectic form $\bar{\omega} = d(r\eta)$ where r is the coordinate on $\mathbb{R}_{>0}$. However, for Sasaki geometry, to ensure compatibility with the extended Riemannian metric, the Kähler form must scale with r^2 rather than r.

Chapter 2

Sasakian manifolds

We have seen that a contact structure on an odd-dimensional manifold M is exactly equivalent to a homogeneous symplectic structure on its cone $C(M)$. If we want to add a Riemannian metric on M that works harmoniously with the contact structure η, we can simply define a Kähler structure on $C(M)$. This leads directly to the definition of a Sasakian manifold as we will show in Section 2.3. However, it is useful to start with a more direct approach to Riemannian metrics on odd-dimensional contact manifolds.

2.1 Almost Contact Metric Structures

Just as an almost complex structure J ($J^2 = -I$) is the topological prerequisite for Kähler geometry, an almost contact structure is the prerequisite for Sasaki geometry.

Definition 2.1.1 (Almost Contact Structure)**.** An **almost contact structure** on an odd-dimensional manifold M^{2n+1} is a triple (φ, ξ, η), where:

- $\varphi \in \mathrm{End}(TM)$ is a tensor field of type $(1,1)$,
- ξ is a vector field,
- η is a 1-form,

satisfying the following algebraic relations:

$$\eta(\xi) = 1, \qquad \varphi^2 = -I + \eta \otimes \xi. \tag{2.1}$$

Notice that on the distribution $H = \ker \eta$, the endomorphism φ satisfies $\varphi^2 = -I$, meaning φ acts as an almost complex structure on the subbundle H.

Proposition 2.1.2. *Any almost contact structure satisfies the following properties:*

1. $\varphi(\xi) = 0$,
2. $\eta \circ \varphi = 0$,
3. $\operatorname{rank} \varphi = 2n$.

Exercise 2.1.3. *Prove Proposition 2.1.2.*

Hint for 1:

1. *Applying φ to (2.1), prove that $\varphi^2(\xi) = 0$,*
2. *Assume $Y := \varphi(\xi) \neq 0$, and proceed by contradiction, keeping in mind that $\varphi(Y) = 0$.*
3. *Notice that 1. implies $Y = \eta(Y)\xi$, hence $0 = \varphi(Y) = \eta(Y)\varphi(\xi) = \eta(Y)Y$.*
4. *So, $\eta(Y) = 0$ and $Y = \eta(Y)\xi = 0$ gives the contradiction.*

Hint for 2:

1. *$\varphi(\varphi^2 X) = \varphi^2(\varphi X) = -\varphi X + \eta(\varphi X)\xi$.*
2. *Using $\varphi(\xi) = 0$, obtain $\eta(\varphi X)\,\xi = 0$.*

We now introduce a Riemannian metric.

Definition 2.1.4 (Contact Metric Manifold)**.** A Riemannian metric g is said to be **compatible** with the almost contact structure (φ, ξ, η) if:

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad \text{for all } X, Y \in \mathfrak{X}(TM). \tag{2.2}$$

The tuple $(M, \varphi, \xi, \eta, g)$ is called an **almost contact metric manifold**. It is not difficult to check that

$$g(\xi, Y) = \eta(Y).$$

Thus η is the metric dual of ξ. Furthermore, M is called a **contact metric manifold** if η is a contact form and the fundamental 2-form

$$\Phi(X, Y) := g(X, \varphi Y)$$

satisfies:

$$d\eta = 2\Phi. \tag{2.3}$$

Exercise 2.1.5. *Prove that $g(\xi, Y) = \eta(Y)$. Moreover, show that the tensor $\Phi(X, Y) := g(X, \varphi Y)$ is skew-symmetric, that is, it defines a 2-form.*

Warning on Conventions!

In differential geometry, conventions regarding wedge products and exterior derivatives often vary. In these notes, we adopt the following approach:

- For **differential forms**, we adopt the convention used by Lee and Spivak, where $d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])$. This introduces a factor of 2 in equations like $d\eta = 2\Phi$ compared to books like Blair or Boyer-Galicki, which use the Kobayashi-Nomizu convention.
- For the **complex structure on the cone** (Section 2.3), we strictly follow the **Boyer-Galicki** convention, defining $J(r\partial_r) = -\xi$.

Remark 2.1.6. Equation (2.2) implies that $\eta(X) = g(X,\xi)$. Thus, **the Reeb vector field is precisely the metric dual of the contact form**. Furthermore, the splitting

$$TM = H \oplus \mathbb{R}\xi$$

is an orthogonal direct sum with respect to g. Equation (2.3) bridges the topological contact structure with the Riemannian geometry: it says that φ relates the metric g to the symplectic form $d\eta$ on H, exactly like an almost Kähler manifold.

2.2 K-Contact Manifolds

In Chapter 1, we saw that the flow of the Reeb vector field ξ always preserves the contact form η, that is, it acts by strict contactomorphisms. However, it does not necessarily preserve the metric g. When it does, we obtain a special class of manifolds.

Definition 2.2.1 (K-Contact Manifold)**.** A contact metric manifold (M,φ,ξ,η,g) is called a **K-contact manifold** if the Reeb vector field ξ is a **Killing vector field** for the metric g. That is:

$$\mathcal{L}_\xi g = 0.$$

This means that the 1-parameter group of diffeomorphisms generated by ξ consists of *isometries* of (M,g).

The following general result will be useful to characterize K-contact manifolds in terms of $\nabla\xi$

Lemma 2.2.2. *Let (M,g) be a Riemannian manifold, and let $Z \in \mathfrak{X}(M)$ be a vector field on M. Then the following formula holds for the Levi-Civita connection*

$$(\mathcal{L}_Z g)(X,Y) = g(\nabla_X Z, Y) + g(X, \nabla_Y Z). \tag{2.4}$$

In particular, the vector field Z is Killing iff the $(1,1)$ tensor field ∇Z is g-skew-symmetric, that is

$$g(\nabla Z(X), Y) = -g(X, \nabla Z(Y)), \tag{2.5}$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Recall that the covariant derivative of a (0,2)-tensor field T is given by the following formula

$$(\nabla_Z T)(X,Y) = Z(T(X,Y)) - T(\nabla_Z X, Y) - T(X, \nabla_Z Y).$$

On the other hand, the Lie derivative is

$$\begin{aligned}(\mathcal{L}_Z T)(X,Y) &= Z(T(X,Y)) - T(\mathcal{L}_Z X, Y) - T(X, \mathcal{L}_Z Y) \\ &= Z(T(X,Y)) - T(\nabla_Z X, Y) + T(\nabla_X Z, Y) - T(X, \nabla_Z Y) + T(X, \nabla_Y Z).\end{aligned}$$

Hence we get the following relation between the Lie derivative and the covariant derivative

$$(\mathcal{L}_Z T)(X,Y) = (\nabla_Z T)(X,Y) + T(\nabla_X Z, Y) + T(X, \nabla_Y Z).$$

If $T = g$ and ∇ is the corresponding Levi-Civita connection, one has $\nabla g = 0$ and thus

$$(\mathcal{L}_Z g)(X,Y) = g(\nabla_X Z, Y) + g(X, \nabla_Y Z).$$

Thus Z is a Killing vector field, that is $(\mathcal{L}_Z g) = 0$, if and only if

$$g(\nabla_X Z, Y) = -g(X, \nabla_Y Z)$$

that is

$$g(\nabla Z(X), Y) = -g(X, \nabla Z(Y)).$$

for arbitrary vector fields X, Y. □

Now we can prove the promised characterization of K-contact manifolds.

Proposition 2.2.3. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. It is K-contact if and only if:*

$$\nabla_X \xi = -\varphi X \quad \text{for all } X \in \mathfrak{X}(M). \tag{2.6}$$

Proof. By Lemma 2.2.2 we have that the Reeb vector field ξ is Killing if and only if the tensor $\nabla\xi$ is skew-symmetric with respect to g, that is, if and only if

$$g(\nabla_X \xi, Y) = -g(X, \nabla_Y \xi), \tag{2.7}$$

for all $X, Y \in \mathfrak{X}(M)$. On any Riemannian manifold, the exterior derivative of a 1-form η (where $\eta(X) = g(X, \xi)$) is related to the Levi-Civita connection by:

$$d\eta(X,Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X).$$

Assume ξ is Killing: Then by substituting (2.7) this into the $d\eta$ equation yields:

$$d\eta(X,Y) = 2g(\nabla_X \xi, Y).$$

However, we are in a contact metric manifold, so by Definition 2.1.4, $d\eta(X,Y) = 2g(X, \varphi Y)$. Thus, since $g(\varphi X, Y) = -g(X, \varphi Y)$ we get

$$-2g(\varphi X, Y) = 2g(X, \varphi Y) = 2g(\nabla_X \xi, Y), \qquad \text{for all } X, Y \in \mathfrak{X}(M).$$

or

$$\nabla_X \xi = -\varphi X, \qquad \text{for all } X \in \mathfrak{X}(M).$$

Conversely: If $\nabla_X \xi = -\varphi X$, then since φ is g-skew-symmetric we have that $\nabla\xi$ satisfies

$$g(\nabla_X \xi, Y) = -g(X, \nabla_Y \xi),$$

meaning ξ is Killing (this implication is true even if the structure is just almost contact metric). □

Proposition 2.2.4. *In a contact metric manifold $(M, \varphi, \xi, \eta, g)$ the integral curves of the Reeb vector field are geodesics, that is,*

$$\nabla_\xi \xi = 0.$$

Exercise 2.2.5. *Prove Proposition 2.2.4.* ***Hint:*** *Use that $\mathcal{L}_\xi \eta = 0$ and that the Reeb vector field has constant norm 1, hence $\nabla_X \xi$ is orthogonal to ξ, for each X.*

2.3 The Metric Cone and Sasakian Structures

We now reach the central definition of the course. How do we rigorously define the odd-dimensional analogue of a Kähler manifold?

Recall from Chapter 1 the **symplectization**: given a contact manifold (M, η), the cone $C(M) = M \times \mathbb{R}_{>0}$ admits a symplectic form ω that is exact and homogeneous of degree 2 (to ensure compatibility with the extended Riemannian metric). We now add the construction of the metric on the cone.

Definition 2.3.1 (The Metric Cone)**.** Let (M, g) be a Riemannian manifold. The **metric cone** over M is the manifold $C(M) = M \times \mathbb{R}_{>0}$ equipped with the warped product metric:

$$G = dr \otimes dr + r^2 g.$$

If M has an almost contact metric structure (φ, ξ, η, g), we can naturally define an **almost complex structure** J on the cone $C(M)$. As $J \in \mathrm{End}(TC(M))$ must be linear we define its action as follows

$$J(X) = \varphi(X) + \eta(X)\left(r\frac{\partial}{\partial r}\right), \qquad \text{for } X \in TM; \tag{2.8}$$

$$J\left(r\frac{\partial}{\partial r}\right) = -\xi. \tag{2.9}$$

Let's verify $J^2 = -I$. For the radial direction:

$$J^2(r\partial_r) = J(-\xi) = -\left(\varphi(\xi) + \eta(\xi) r\partial_r\right) = 0 - 1(r\partial_r) = -r\partial_r.$$

For a vector X tangent to M:

$$J^2(X) = J\left(\varphi(X) + \eta(X) r\partial_r\right) = J(\varphi X) + \eta(X)(-\xi) = \varphi^2(X) + \eta(\varphi X) r\partial_r - \eta(X)\xi.$$

Since $\eta \circ \varphi = 0$ and $\varphi^2 X = -X + \eta(X)\xi$, this simplifies to:

$$J^2(X) = -X + \eta(X)\xi + 0 - \eta(X)\xi = -X.$$

Thus, J is indeed an almost complex structure on the metric cone $C(M)$. Furthermore, the compatible Kähler form ω on $C(M)$ is given by

$$\omega = \frac{1}{2} d(r^2 \eta) = \frac{1}{2} r^2 d\eta + r dr \wedge \eta.$$

(Note that this formula adheres to the convention that $\omega(A, B) = G(A, JB)$).

Definition 2.3.2 (Sasakian Manifold)**.** A contact metric manifold $(M, \varphi, \xi, \eta, g)$ is a **Sasakian manifold** if its metric cone $(C(M), G, J)$ is a **Kähler manifold**.

This definition is geometrically beautiful. It states that Sasakian manifolds are exactly the cross-sections of Kähler cones. If $C(M)$ is Calabi-Yau (Ricci-flat Kähler), then M is called a *Sasaki-Einstein* manifold.

2.4 Equivalent Definitions of Sasakian Structures

While the cone definition is geometrically elegant, it is often difficult to work with computationally because you have to move up a dimension. Fortunately, there are equivalent formulations entirely on M.

2.4.1 The Normality Condition

For $(C(M), G, J)$ to be Kähler, J must be integrable (i.e., it must arise from a true complex manifold). By the Newlander-Nirenberg theorem, J is integrable if and only if its **Nijenhuis tensor** N_J vanishes identically.

$$N_J(A, B) = [JA, JB] - [A, B] - J[JA, B] - J[A, JB] = 0.$$

Pushing this integrability condition down to M via the projection $C(M) \to M$ gives the so-called **normality condition**.

Definition 2.4.1 (Normality)**.** An almost contact structure (φ, ξ, η) is **normal** if the following tensor $N^{(1)}$ defined on M vanishes identically:

$$N^{(1)}(X, Y) := N_\varphi(X, Y) + d\eta(X, Y) \otimes \xi = 0, \tag{2.10}$$

where $N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$ is the Nijenhuis tensor of the endomorphism φ.

Theorem 2.4.2. *A contact metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasakian if and only if it is normal.*

Proof sketch: A direct but tedious calculation of N_J on the cone evaluated on vector fields lifted from M reveals that the components of N_J are exactly linear combinations of $N^{(1)}$ and its contractions. Thus $N_J \equiv 0 \iff N^{(1)} \equiv 0$.

Exercise 2.4.3. *Fill in the details of the proof of Theorem 2.4.2.* ***Hint:*** *Use that $\mathcal{L}_\xi \eta = 0$ and that the Reeb vector field has constant norm 1, hence $\nabla_X \xi$ is orthogonal to ξ, for each X.*

2.4.2 The Levi-Civita Characterization

The most practical definition for performing differential geometry on a Sasakian manifold uses the Levi-Civita connection ∇ of the metric g on M.

Theorem 2.4.4. *A contact metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasakian if and only if:*

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.11}$$

for all vector fields $X, Y \in \Gamma(TM)$.

This is the Sasakian analogue of the Kähler condition $\nabla J = 0$. In Sasaki geometry, φ is *not* parallel. Its covariant derivative is completely determined by the metric and the contact form.

Proposition 2.4.5. *Every Sasakian manifold is K-contact.*

Proof. Let M be Sasakian. By Theorem 2.4.4, we have

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

Let us evaluate this identity for $Y = \xi$. Since $\varphi(\xi) = 0$, the left-hand side becomes:

$$(\nabla_X \varphi)\xi = \nabla_X(\varphi\xi) - \varphi(\nabla_X \xi) = -\varphi(\nabla_X \xi).$$

The right-hand side, since $\eta(\xi) = 1$, becomes:

$$g(X,\xi)\xi - \eta(\xi)X = \eta(X)\xi - X.$$

Thus,

$$-\varphi(\nabla_X \xi) = \eta(X)\xi - X.$$

Apply φ to both sides. Recall $\varphi^2(Z) = -Z + \eta(Z)\xi$.

$$-\varphi^2(\nabla_X \xi) = \varphi(\eta(X)\xi) - \varphi(X) \implies -(-\nabla_X \xi + \eta(\nabla_X \xi)\xi) = 0 - \varphi(X).$$

Since $\eta(\nabla_X \xi) = g(\nabla_X \xi, \xi) = \frac{1}{2}X(g(\xi,\xi)) = \frac{1}{2}X(1) = 0$, we are left exactly with:

$$\nabla_X \xi = -\varphi X.$$

By Proposition 2.2.3, this means ξ is a Killing vector field, hence M is K-contact. □

Remark 2.4.6. The converse is **false** in general. Every Sasakian manifold is K-contact, but not every K-contact manifold is Sasakian. Though remarkably, in dimension 3, K-contact implies Sasakian. The discrepancy begins in dimension 5 and higher. In Chapter 4, the Hard Lefschetz theorem will give us the topological tools to definitively prove that certain K-contact manifolds cannot be Sasakian.

2.5 Example: The Standard Sphere

In Chapter 1 (Example 1.4.2), we defined a contact form on $S^{2n+1} \subset \mathbb{C}^{n+1}$:

$$\eta = \sum_{k=1}^{n+1} (y_k\, dx_k - x_k\, dy_k).$$

Let g be the standard round metric on S^{2n+1} induced by the flat Euclidean metric on $\mathbb{C}^{n+1}$. The metric cone $C(S^{2n+1}) = S^{2n+1} \times \mathbb{R}_{>0}$ over the unit sphere (S^{2n+1}, g) with the standard round metric g is exactly $\mathbb{R}^{2n+2} \setminus \{0\} \cong \mathbb{C}^{n+1} \setminus \{0\}$, with the metric

$$G = dr \otimes dr + r^2 g.$$

Actually G happens to be just the flat Euclidean metric written in hyperspherical (polar) coordinates. We know that $\mathbb{C}^{n+1}$ and hence $\mathbb{C}^{n+1} \setminus \{0\}$ with the flat metric and the standard integrable complex structure J_0 is Kähler. By Definition 2.3.2, since the metric cone $C(S^{2n+1}) \cong \mathbb{C}^{n+1} \setminus \{0\}$ is a Kähler manifold, the induced structure (φ, ξ, η, g) on S^{2n+1} is automatically a **Sasakian structure**.

This makes the odd-dimensional spheres the canonical, foundational examples of Sasaki geometry, much like $\mathbb{C}P^n$ is for Kähler geometry.

Exercise 2.5.1. *Verify that the metric cone $C(S^{2n+1})$ is a Kähler manifold.*

Chapter 3

Transverse geometry

A defining feature of Sasakian geometry is that it behaves locally like Kähler geometry in the directions transverse to the Reeb vector field. To make this precise, we must understand the foliation generated by the Reeb flow and develop tools to study its transverse space, even when the global quotient space is highly singular.

3.1 The Reeb Foliation

Recall that on a manifold M of dimension $2n+1$, a vector field that is nowhere zero defines a 1-dimensional *foliation*. The orbits (or integral curves) of the vector field are called the *leaves* of the foliation.

Definition 3.1.1 (The Reeb Foliation)**.** Let (M, η) be a contact manifold with Reeb vector field ξ. The **Reeb foliation**, denoted $\mathcal{F}_\xi$, is the 1-dimensional foliation of M generated by the flow of ξ. The tangent bundle to this foliation is the line bundle $L_\xi = \mathbb{R}\xi \subset TM$.

The global topological behavior of the leaves deeply influences how we can study the transverse geometry. Depending on the action generated by the Reeb flow, Sasaki manifolds are classified into three distinct categories:

Definition 3.1.2 (Regularity of Sasaki Manifolds)**.** A Sasakian manifold M is called:

- **Regular:** If the flow of ξ integrates to a free circle action $S^1 \times M \to M$. In this case, all Reeb orbits are closed and have the same period.
- **Quasi-regular:** If the flow of ξ integrates to a locally free S^1-action. All orbits are closed, but they may have different periods.
- **Irregular:** If the flow of ξ does not integrate to an S^1-action. In this case, at least some Reeb orbits are not closed (they can be dense in higher-dimensional tori $\mathbb{T}^k$).

3.1.1 The Regular Case: Boothby-Wang Fibration

When the Reeb foliation is regular, the geometry of M can be perfectly described in terms of a smooth quotient manifold. This is the content of a classical theorem.

Theorem 3.1.3 (Boothby-Wang Theorem, 1958)**.** *Let (M, η) be a compact regular contact manifold. Then the space of leaves $B = M/\mathcal{F}_\xi$ is a smooth manifold of dimension $2n$, and the canonical projection $\pi : M \to B$ is a principal S^1-bundle.*

Furthermore, B admits a unique symplectic form ω such that $\pi^*\omega = d\eta$. In this picture:

- *The contact form η acts as a principal connection 1-form on $M \xrightarrow{\pi} B$.*
- *The 2-form $d\eta$ is the curvature of this connection.*

*If M carries a Sasakian structure (φ, ξ, η, g), then the base space (B, ω) naturally inherits a **Kähler structure**.*

The Boothby-Wang fibration is the prototype for understanding Sasaki geometry. Every regular Sasaki manifold is a circle bundle over a Kähler manifold (e.g., the Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$). If M is quasi-regular, the quotient $M/\mathcal{F}_\xi$ is not a smooth manifold but a Kähler *orbifold*.

However, if M is **irregular**, the space of leaves $M/\mathcal{F}_\xi$ is a highly singular, non-Hausdorff topological space. How do we do differential geometry on a space that doesn't exist smoothly? **We do it on M itself, using a specialized sub-algebra of differential forms.**

3.2 Basic forms and Basic Cohomology

To study the "transverse space" even when the quotient is pathological, we single out the differential forms on M that "think" they live on the quotient space $M/\mathcal{F}_\xi$.

Definition 3.2.1 (Basic Form)**.** Let (M, η) be a contact manifold with Reeb field ξ. A differential p-form $\alpha \in \Omega^p(M)$ is called **basic** with respect to $\mathcal{F}_\xi$ if it satisfies two conditions:

1. $i_\xi \alpha = 0$ (It has no component in the Reeb direction).
2. $\mathcal{L}_\xi \alpha = 0$ (It is invariant under the Reeb flow).

The space of all basic p-forms is denoted $\Omega^p_B(M)$.

Remark 3.2.2. In the regular case (Boothby-Wang), a form $\alpha \in \Omega^p(M)$ is basic if and only if it is the pullback of a form on the base space B. That is, $\Omega^p_B(M) = \pi^*\Omega^p(B)$. The concept of basic forms brilliantly generalizes the forms on the quotient to cases where the quotient does not smoothly exist.

Exercise 3.2.3. *Prove that in the regular case each basic p-form on the total space M of the bundle $\pi\colon M \to B$ is the pullback of a p-form on the base manifold B.*

Proposition 3.2.4. *The set of basic forms $\Omega^*_B(M) = \bigoplus_{p=0}^{2n} \Omega^p_B(M)$ forms a graded commutative sub-algebra of $\Omega^*(M)$ that is preserved by the exterior derivative d.*

Proof. Let $\alpha \in \Omega_B^p$ and $\beta \in \Omega_B^q$. First, we check stability under the wedge product.

$$i_\xi(\alpha \wedge \beta) = (i_\xi \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_\xi \beta) = 0 + 0 = 0.$$
$$\mathcal{L}_\xi(\alpha \wedge \beta) = (\mathcal{L}_\xi \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_\xi \beta) = 0 + 0 = 0.$$

Thus, $\Omega_B^*(M)$ is a subalgebra. Second, we check stability under the exterior derivative d. Let $\alpha \in \Omega_B^p$. We need to show $d\alpha \in \Omega_B^{p+1}$.

1. By Cartan's magic formula, $\mathcal{L}_\xi = d \circ i_\xi + i_\xi \circ d$. Applying this to α:

$$0 = \mathcal{L}_\xi \alpha = d(i_\xi \alpha) + i_\xi(d\alpha) = d(0) + i_\xi(d\alpha) \implies i_\xi(d\alpha) = 0.$$

2. Because the Lie derivative commutes with the exterior derivative:

$$\mathcal{L}_\xi(d\alpha) = d(\mathcal{L}_\xi \alpha) = d(0) = 0.$$

Therefore, $d\alpha$ satisfies both conditions, so $d(\Omega_B^p) \subset \Omega_B^{p+1}$. □

Example 3.2.5 (Functions and Constants)**.** A basic 0-form (a function) $f \in \Omega_B^0(M)$ trivially satisfies $i_\xi f = 0$. The second condition $\mathcal{L}_\xi f = \xi(f) = df(\xi) = 0$ means that f is constant along the Reeb orbits. If the Reeb orbits are dense in M (the extremely irregular case), the only basic functions are global constants!

Example 3.2.6 (The 2-form $d\eta$)**.** Is the contact form η basic? No, because $i_\xi \eta = 1 \neq 0$. However, consider $d\eta$. By definition of the Reeb vector field, $i_\xi d\eta = 0$. By Proposition 1.5.2, $\mathcal{L}_\xi d\eta = 0$. Therefore, $d\eta$ **is a basic 2-form**. In fact, we will see it plays the role of the Kähler form for the transverse space.

3.2.1 Basic Cohomology

Since the exterior derivative d maps basic forms to basic forms (Proposition 3.2.4), and $d^2 = 0$, we get that the basic forms define a chain complex.

$$0 \to \Omega_B^0(M) \xrightarrow{d_B} \Omega_B^1(M) \xrightarrow{d_B} \dots \xrightarrow{d_B} \Omega_B^{2n}(M) \to 0,$$

where d_B is simply the restriction of the usual de Rham differential d to $\Omega_B^*(M)$.

Definition 3.2.7 (Basic Cohomology)**.** The **basic cohomology groups** of the Reeb foliation $\mathcal{F}_\xi$ are defined as:

$$H_B^k(M) = \frac{\ker(d_B : \Omega_B^k \to \Omega_B^{k+1})}{\mathrm{Im}(d_B : \Omega_B^{k-1} \to \Omega_B^k)}.$$

If M is regular with smooth leaf space B, the basic cohomology is canonically isomorphic to the de Rham cohomology of the base: $H_B^*(M) \cong H_{dR}^*(B)$. However, $H_B^*(M)$ is perfectly well-defined even when the leaf space is pathological.

Remark 3.2.8. A deep theorem by El Kacimi-Alaoui (1990) states that for Riemannian foliations on compact manifolds (and the Reeb foliation on a K-contact manifold is a prime example), the basic cohomology groups $H_B^k(M)$ are **finite-dimensional** and satisfy a version of Poincaré duality.

3.3 Transverse Kähler Geometry

Now we connect the basic forms to the metric structure of a Sasakian manifold $(M, \varphi, \xi, \eta, g)$.

The tangent bundle naturally splits as $TM = H \oplus \mathbb{R}\xi$, where $H = \ker \eta$ is the contact distribution. Let $\pi_H : TM \to H$ be the orthogonal projection onto H. Because ξ is a Killing vector field for g (since Sasakian implies K-contact), the metric g restricted to H is invariant under the Reeb flow. We call $g^T = g|_H$ the **transverse metric**.

Similarly, we know $\varphi(\xi) = 0$ and $\varphi(H) \subset H$. Furthermore, $\varphi^2|_H = -I$. Thus, φ restricts to an almost complex structure on H, which we denote by $J^T = \varphi|_H$.

3.3.1 The Transverse Levi-Civita Connection

To do Riemannian geometry on the transverse space H, we need a connection. The standard Levi-Civita connection ∇ of M doesn't exactly preserve H. We must project it.

Definition 3.3.1 (Transverse Connection)**.** The **transverse Levi-Civita connection** ∇^T on the subbundle H is defined by:

$$\nabla^T_X Y = \pi_H(\nabla_X Y), \quad \text{for } X, Y \in \Gamma(H).$$

We can also extend ∇^T allowing to derive sections of H also along the ξ direction using the Lie derivative, by setting

$$\nabla^T_\xi Y = \pi_H(\mathcal{L}_\xi Y) = \pi_H([\xi, Y]), \quad \text{for } Y \in \Gamma(H).$$

In this way we obtain a connection on H (treated as a vector bundle over M) that fully captures the transverse geometry.

3.3.2 Transverse Kähler structure of Sasakian manifolds

We are now ready to state and prove an important geometric insight of Sasaki geometry.

Theorem 3.3.2 (Sasakian implies Transverse Kähler)**.** *Let $(M, \varphi, \xi, \eta, g)$ be a K-contact manifold. Then M is Sasakian if and only if the transverse almost complex structure J^T is integrable and $\nabla^T J^T = 0$. Furthermore, the basic 2-form $\Phi = \frac{1}{2} d\eta$ acts as the transverse Kähler form, and it is basic-closed ($d_B \Phi = 0$).*

Proof. **(Optional)** Let us analyze the condition $\nabla^T J^T = 0$. For any $X, Y \in \Gamma(H)$, we compute:

$$(\nabla^T_X J^T)Y = \nabla^T_X(J^T Y) - J^T(\nabla^T_X Y) = \pi_H \nabla_X(\varphi Y) - \varphi(\pi_H \nabla_X Y).$$

Because φ commutes with the projection π_H (since $\varphi(\xi) = 0$ and $\varphi(H) \subset H$), we have:

$$(\nabla^T_X J^T)Y = \pi_H(\nabla_X(\varphi Y) - \varphi(\nabla_X Y)) = \pi_H((\nabla_X \varphi)Y).$$

Recall from Theorem 2.4.4 that M is Sasakian if and only if:

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

Because $X, Y \in H$, we have $\eta(Y) = 0$. Thus the Sasaki condition becomes:

$$(\nabla_X \varphi)Y = g(X, Y)\xi.$$

Notice that the right-hand side is entirely in the $\mathbb{R}\xi$ direction. Therefore, if M is Sasakian, taking the projection onto H yields:

$$(\nabla^T_X J^T)Y = \pi_H(g(X,Y)\xi) = 0.$$

Conversely, if $\pi_H((\nabla_X\varphi)Y) = 0$, then $(\nabla_X\varphi)Y$ must be a multiple of ξ. Careful analysis using the contact metric identities forces that multiple to exactly be $g(X,Y)\xi$, recovering the Sasaki condition.

What about the Kähler form? The fundamental 2-form of the transverse structure is $g^T(X, J^TY) = g(X,\varphi Y) = \frac{1}{2}d\eta(X,Y)$. As proven in Example 3.2.6, $\Phi \propto d\eta$ is a basic 2-form. Since it is exact in the full de Rham complex, its basic exterior derivative is obviously zero ($d_B\Phi = 0$). □

Remark 3.3.3 (The Kähler Class)**.** Even though $d\eta$ is an exact 2-form on M ($d\eta = d(\eta)$), the primitive η is **not** a basic form (since $i_\xi\eta = 1$). Indeed, we will now show that, within the basic complex, $d\eta$ is **closed but not exact**! We say a basic form is *basic-exact* if it can be written as the exterior derivative of another basic form.

Lemma 3.3.4. *Let (M^{2n+1}, η) be a compact, closed contact manifold with Reeb vector field ξ. The differential of the contact form, $d\eta$, cannot be exact in the basic complex $\Omega^*_B(M, \mathcal{F}_\xi)$. That is, its basic cohomology class $[d\eta]_B \in H^2_B(M, \mathcal{F}_\xi)$ is non-trivial.*

Proof. First, we verify that $d\eta$ is indeed a basic 2-form. By the definition of the Reeb vector field, we have $i_\xi(d\eta) = 0$. Using Cartan's magic formula, the Lie derivative is given by

$$\mathcal{L}_\xi(d\eta) = d(i_\xi d\eta) + i_\xi(d(d\eta)) = d(0) + i_\xi(0) = 0.$$

Thus, $d\eta \in \Omega^2_B(M)$.

We proceed by contradiction. Assume that $d\eta$ is basic-exact. This means there exists a basic 1-form $\beta \in \Omega^1_B(M)$ such that

$$d\eta = d\beta.$$

Because M is a compact contact manifold, integrating the canonical volume form yields a strictly positive total volume:

$$\int_M \eta \wedge (d\eta)^n > 0. \tag{3.1}$$

Let us substitute $d\eta = d\beta$ into the first factor of $(d\eta)^n$ in the volume form:

$$\eta \wedge (d\eta)^n = \eta \wedge d\beta \wedge (d\eta)^{n-1}.$$

Since $d(d\eta) = 0$, we can pull the exterior derivative outside the wedge product of the basic forms:

$$\eta \wedge d\beta \wedge (d\eta)^{n-1} = \eta \wedge d\big(\beta \wedge (d\eta)^{n-1}\big).$$

Define $\gamma = \beta \wedge (d\eta)^{n-1}$. Because the wedge product of basic forms is basic, γ is a basic $(2n-1)$-form, which implies $i_\xi\gamma = 0$. We now have:

$$\eta \wedge (d\eta)^n = \eta \wedge d\gamma.$$

By the Leibniz rule for exterior derivatives, we know that

$$d(\eta \wedge \gamma) = d\eta \wedge \gamma - \eta \wedge d\gamma.$$

Rearranging this equation gives us an expression for our volume form:

$$\eta \wedge d\gamma = d\eta \wedge \gamma - d(\eta \wedge \gamma). \tag{3.2}$$

Now, let us analyze the term $d\eta \wedge \gamma$. Expanding γ, we get:

$$d\eta \wedge \gamma = d\eta \wedge \beta \wedge (d\eta)^{n-1} = \beta \wedge (d\eta)^n.$$

Let $\omega = \beta \wedge (d\eta)^n$. Since both β and $(d\eta)^n$ are basic forms, ω is a basic $(2n+1)$-form. Because it is basic, it must satisfy $i_\xi \omega = 0$.

However, M has dimension $2n+1$. Any $(2n+1)$-form on a $(2n+1)$-dimensional manifold is a top-degree form, which means it is everywhere proportional to the volume form. Since the Reeb vector field ξ is nowhere-vanishing (as $i_\xi \eta = 1$), the contraction $i_\xi \omega$ of any non-zero top-degree form ω cannot be identically zero. Therefore, to satisfy $i_\xi \omega = 0$, the top form ω must be identically zero on M. Thus:

$$d\eta \wedge \gamma = 0.$$

Substituting this back into Equation (3.2), we obtain:

$$\eta \wedge (d\eta)^n = -d(\eta \wedge \gamma).$$

We now integrate both sides over the compact manifold M:

$$\int_M \eta \wedge (d\eta)^n = -\int_M d(\eta \wedge \gamma).$$

Since M is a compact manifold without boundary, Stokes' Theorem dictates that the integral of any globally exact form is zero:

$$\int_M d(\eta \wedge \gamma) = 0.$$

This implies:

$$\int_M \eta \wedge (d\eta)^n = 0.$$

This contradicts Equation (3.1), which states that the integral of the volume form must be strictly positive.

Consequently, our assumption must be false. The 2-form $d\eta$ cannot be basic-exact. □

Remark. Because $d\eta$ is a closed basic form but not basic-exact, it defines a non-trivial cohomology class $[d\eta]_B \in H^2_B(M)$. In Sasakian geometry, this fundamental invariant is known as the *transverse Kähler class*. It defines a non-trivial cohomology class $[d\eta]_B \in H^2_B(M)$, which acts exactly like the Kähler class of a compact Kähler manifold.

Remark 3.3.5. Necessity of Compactness: The compactness of M is an essential hypothesis. On a non-compact contact manifold, such as $\mathbb{R}^{2n+1}$ equipped with the standard contact form $\eta = dz - \sum_{i=1}^n y_i dx_i$, the differential $d\eta = \sum dx_i \wedge dy_i$ is basic and exact in the basic complex (e.g., taking the basic 1-form $\beta = \sum x_i dy_i$, we have $d\beta = d\eta$).

3.4 The Transverse Hard Lefschetz Theorem

In Chapter 2, we defined the fundamental 2-form $\Phi(X,Y) = g(X,\phi Y)$. Because we are on a contact metric manifold, $d\eta(X,Y) = 2\Phi(X,Y)$ (using our conventions). Because $i_\xi d\eta = 0$ and $\mathcal{L}_\xi d\eta = 0$, $d\eta$ is a basic 2-form. Furthermore, since $d(d\eta) = 0$, it represents a basic cohomology class:

$$[d\eta]_B \in H_B^2(M).$$

This class $[d\eta]_B$ plays the exact role of the Kähler class in the transverse geometry. Let us define the **transverse Lefschetz operator** $L_B : \Omega_B^p \to \Omega_B^{p+2}$ by:

$$L_B(\alpha) = d\eta \wedge \alpha.$$

Assume that M is Sasakian. Because it is transversally Kähler, one can apply transverse Hodge theory (due to El Kacimi-Alaoui) to show that the standard local commutation relations of Kähler geometry hold for the basic operators. Then, by applying finite-dimensional representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, one obtains the following profound result similar to the Hard Lefschetz Theorem that holds for a compact Kähler manifold. The difference is that here the result holds for the basic cohomology.

Theorem 3.4.1 (Transverse Hard Lefschetz Theorem)**.** *Let M be a compact Sasakian manifold of dimension $2n+1$. For every $k \leq n$, the map defined by taking the k-th wedge power of $d\eta$,*

$$L_B^k : H_B^{n-k}(M) \longrightarrow H_B^{n+k}(M), \quad [\alpha]_B \longmapsto [(d\eta)^k \wedge \alpha]_B,$$

*is an **isomorphism**.*

El Kacimi-Alaoui also showed that a transverse Poincaré duality holds.

Theorem 3.4.2 (Transverse Poincaré Duality)**.** *The top basic cohomology group is $H_B^{2n}(M) \cong \mathbb{R}$. Furthermore, the bilinear pairing $H_B^k(M) \times H_B^{2n-k}(M) \to \mathbb{R}$ given by*

$$\langle [\alpha], [\beta] \rangle = \int_M \alpha \wedge \beta \wedge \eta$$

is non-degenerate.

As a consequence we get some information on the **basic** Betti numbers. first of all we have that the even and odd basic Betti number increase monotonically.

Moreover, we get the following result.

Theorem 3.4.3. *Let M be a compact Sasakian manifold of dimension $2n+1$. For any odd integer, the basic Betti number b_k^B is even.*

Proof. By Transverse Poincaré Duality, $b_k^B = b_{2n-k}^B$. Therefore, it suffices to prove the statement for odd $k \leq n$.

Let $k < n$ be an odd integer. We define a bilinear form $Q : H_B^k(M,\mathbb{R}) \times H_B^k(M,\mathbb{R}) \to \mathbb{R}$ by composing the Transversal Hard Lefschetz isomorphism with the Transverse Poincaré Duality pairing. Explicitly, we set:

$$Q([\alpha],[\beta]) = \langle [\alpha], L_B^{n-k}([\beta]) \rangle = \int_M \alpha \wedge \beta \wedge (d\eta)^{n-k} \wedge \eta.$$

We will show that Q is both non-degenerate and skew-symmetric.

1. Non-degeneracy: Suppose there exists a class $[\alpha] \in H_B^k(M, \mathbb{R})$ such that $Q([\alpha], [\beta]) = 0$ for all $[\beta] \in H_B^k(M, \mathbb{R})$. This implies that

$$\langle [\alpha], L_B^{n-k}([\beta]) \rangle = 0 \quad \text{for all } [\beta] \in H_B^k(M, \mathbb{R}).$$

Because the map $L_B^{n-k} : H_B^k(M, \mathbb{R}) \to H_B^{2n-k}(M, \mathbb{R})$ is an isomorphism, the elements $L_B^{n-k}([\beta])$ span the entirety of $H_B^{2n-k}(M, \mathbb{R})$. Since the Transverse Poincaré Duality pairing $\langle \cdot, \cdot \rangle$ is non-degenerate, it must follow that $[\alpha] = 0$. Hence, Q is non-degenerate.

2. Skew-symmetry: Let $[\alpha], [\beta] \in H_B^k(M, \mathbb{R})$. Because k is an odd integer, the wedge product of α and β is anti-commutative:

$$\alpha \wedge \beta = (-1)^{k \cdot k} \beta \wedge \alpha = -\beta \wedge \alpha.$$

The form $(d\eta)^{n-k}$ has degree $2(n-k)$, which is even, and thus commutes with α and β without any sign change. Evaluating $Q([\beta], [\alpha])$, we have:

$$\begin{aligned} Q([\beta], [\alpha]) &= \int_M \beta \wedge \alpha \wedge (d\eta)^{n-k} \wedge \eta \\ &= \int_M (-\alpha \wedge \beta) \wedge (d\eta)^{n-k} \wedge \eta \\ &= -\int_M \alpha \wedge \beta \wedge (d\eta)^{n-k} \wedge \eta \\ &= -Q([\alpha], [\beta]). \end{aligned}$$

Thus, Q is a skew-symmetric bilinear form.

Conclusion: We have constructed a non-degenerate, skew-symmetric bilinear form on the finite-dimensional vector space $H_B^k(M, \mathbb{R})$. It is a standard result in linear algebra that any finite-dimensional vector space admitting such a form must have an even dimension. Therefore, $b_k^B = \dim_{\mathbb{R}} H_B^k(M, \mathbb{R})$ is even. □

Remark 3.4.4. As K-contact manifolds are transversally only almost Kähler, Transverse Hard Lefschetz Theorem generally fails for K-contact manifolds. There is a very restrictive subclass of non-Sasakian manifolds—dubbed "Lefschetz K-contact manifolds"—that do miraculously satisfy the property, but these are constructed as special exceptions rather than the rule.

3.5 The Gysin Sequence

We now know everything about the basic cohomology $H_B^*(M)$. But our ultimate goal is to understand the actual de Rham cohomology $H_{dR}^*(M)$. We need a tool to link them.

3.5.1 The Differential Complexes

To understand the topology of M in relation to the Reeb foliation, we will use two specific subcomplexes of the standard de Rham complex $(\Omega^*(M), d)$.

Definition 3.5.1 (Invariant Forms)**.** The space of invariant k-forms is defined as:

$$\Omega^k_{\text{inv}}(M) = \{\alpha \in \Omega^k(M) \mid \mathcal{L}_\xi \alpha = 0\}.$$

Since the exterior derivative d commutes with the Lie derivative $\mathcal{L}_\xi$, the space $(\Omega^*_{\text{inv}}(M), d)$ forms a subcomplex of the de Rham complex.

Because the Reeb flow on a compact K-contact manifold generates an isometry, So it is contained in the isometry group that is a compact (finite-dimensional) Lie group by the Myers-Steenrod Theorem. By the standard principles of Lie theory, the closure of a 1-parameter subgroup in a compact Lie group is a closed, connected, abelian subgroup. Any such group is isomorphic to a compact torus $\mathbb{T}^k$. By averaging any differential form over this torus, one can construct a projection from $\Omega^*(M)$ to $\Omega^*_{\text{inv}}(M)$ that commutes with d. The following standard result tells us that the inclusion of invariant forms induces an isomorphism in cohomology.

Lemma 3.5.2. *For any compact K-contact manifold, the inclusion of the complex of invariant forms into the full de Rham complex induces an isomorphism in cohomology:*

$$H^*(\Omega^*_{\text{inv}}(M), d) \cong H^*_{dR}(M).$$

The second subcomplex that we will need is the one of basic forms that we already introduced.

Definition 3.5.3 (Basic Forms)**.** Recall that the space of basic k-forms is defined as:

$$\Omega^k_B(M) = \{\alpha \in \Omega^k(M) \mid i_\xi \alpha = 0 \text{ and } \mathcal{L}_\xi \alpha = 0\}.$$

Basic forms are simply invariant forms that are strictly "horizontal" with respect to the Reeb vector field. The cohomology of this subcomplex, denoted by $H^*_B(M)$, is called the **basic cohomology** of the foliation.

3.5.2 The Short Exact Sequence

We are now ready to state and prove our main proposition. We want to link the basic complex to the invariant complex.

Proposition 3.5.4. *For any compact K-contact manifold M, there exists a short exact sequence of graded vector spaces:*

$$0 \longrightarrow \Omega^k_B(M) \xrightarrow{\ i\ } \Omega^k_{\text{inv}}(M) \xrightarrow{\ j\ } \Omega^{k-1}_B(M) \longrightarrow 0 \tag{3.3}$$

where i is the natural inclusion map, and $j(\alpha) = i_\xi \alpha$ is the interior product with the Reeb vector field.

Proof. Let us verify the exactness at each step. This is a classic "diagram chasing" style argument, though heavily rooted in our geometric definitions.

Step 1: Injectivity of i. The map i is merely an inclusion of a subspace, so it is trivially injective.

Step 2: Exactness in the middle ($\ker j = \operatorname{Im} i$). Take any form $\alpha \in \Omega^k_{\text{inv}}(M)$. By definition, we know $\mathcal{L}_\xi \alpha = 0$. Suppose $\alpha \in \ker j$. This means $j(\alpha) = i_\xi \alpha = 0$. Thus, α satisfies both $\mathcal{L}_\xi \alpha = 0$ and $i_\xi \alpha = 0$. But this is precisely the definition of a basic form! Hence, $\alpha \in \Omega^k_B(M)$, which means $\ker j \subseteq \operatorname{Im} i$. The reverse inclusion ($\operatorname{Im} i \subseteq \ker j$) is immediate because if α is basic, $i_\xi \alpha = 0$ by definition. Therefore, exactness holds.

Step 3: Surjectivity of j. This is where we use the contact form η. Let $\gamma \in \Omega^{k-1}_B(M)$. We must construct an invariant k-form α such that $j(\alpha) = \gamma$.

Let us propose the form:

$$\alpha = \eta \wedge \gamma.$$

First, we must ensure α lives in $\Omega^k_{\text{inv}}(M)$. Let's compute its Lie derivative using the Leibniz rule:

$$\mathcal{L}_\xi(\eta \wedge \gamma) = (\mathcal{L}_\xi \eta) \wedge \gamma + \eta \wedge (\mathcal{L}_\xi \gamma).$$

Since η is invariant ($\mathcal{L}_\xi \eta = 0$) and γ is basic (hence invariant, $\mathcal{L}_\xi \gamma = 0$), both terms vanish. Thus, $\mathcal{L}_\xi \alpha = 0$, and α is indeed an invariant form.

Next, we apply j to α. Using the anti-derivation property of the interior product:

$$j(\alpha) = i_\xi(\eta \wedge \gamma) = (i_\xi \eta)\gamma - \eta \wedge (i_\xi \gamma).$$

Recall our initial identities: $i_\xi \eta = 1$ and, since γ is basic, $i_\xi \gamma = 0$. Substituting these in, we obtain:

$$j(\alpha) = 1 \cdot \gamma - \eta \wedge 0 = \gamma.$$

This proves that j is surjective, concluding the proof of the proposition. □

3.5.3 Chain Maps and the Long Exact Sequence

To extract a long exact sequence in cohomology (via the Zig-Zag or Snake Lemma), our maps i and j must commute (or anti-commute) with the exterior derivative d.

Lemma 3.5.5. *The maps i and j interact with the differential d as follows:*

$$d \circ i = i \circ d \quad \textit{and} \quad j \circ d = -d \circ j.$$

Proof. For the inclusion i, it is trivially true since d is the standard exterior derivative on both spaces.

For j, we invoke **Cartan's magic formula**: $\mathcal{L}_\xi = d \circ i_\xi + i_\xi \circ d$. Let $\alpha \in \Omega^k_{\text{inv}}(M)$. Since α is invariant, $\mathcal{L}_\xi \alpha = 0$. Therefore:

$$0 = d(i_\xi \alpha) + i_\xi(d\alpha) \implies i_\xi(d\alpha) = -d(i_\xi \alpha).$$

Rewriting this using our map $j = i_\xi$, we get exactly $j(d\alpha) = -d(j(\alpha))$. □

The anti-commutativity is perfectly fine for homological algebra. We can formally view j as a strict chain map by equipping the target complex $\Omega^{*-1}_B(M)$ with the differential $-d$.

3.5.4 Applying the Snake Lemma (The Gysin Sequence)

By passing to cohomology, the short exact sequence of complexes yields a long exact sequence. Replacing $H^k_{\text{inv}}(M)$ with $H^k_{\text{dR}}(M)$ thanks to the compact torus action mentioned before (Lemma 3.5.2), we arrive at our crowning result.

Theorem 3.5.6 (The Gysin Sequence for K-contact Manifolds)**.** *Let M be a compact K-contact manifold. There is a long exact sequence relating basic cohomology to standard de Rham cohomology:*

$$\dots \longrightarrow H^k_B(M) \xrightarrow{i^*} H^k_{dR}(M) \xrightarrow{j^*} H^{k-1}_B(M) \xrightarrow{\delta} H^{k+1}_B(M) \longrightarrow \dots \tag{3.4}$$

Note: The exact sequence holds for K-contact manifolds, and therefore for all Sasakian manifolds.

3.5.5 Calculating the Connecting Homomorphism δ

To fully understand this sequence geometrically, we must explicitly calculate the connecting homomorphism $\delta : H^{k-1}_B(M) \to H^{k+1}_B(M)$. This is one of those highly instructive calculations in differential geometry.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle -d} & & \\
0 & \longrightarrow & \Omega^k_B(M) & \xrightarrow{i} & \Omega^k_{\text{inv}}(M) & \xrightarrow{j} & \Omega^{k-1}_B(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle -d} & & \\
0 & \longrightarrow & \Omega^{k+1}_B(M) & \xrightarrow{i} & \Omega^{k+1}_{\text{inv}}(M) & \xrightarrow{j} & \Omega^k_B(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle -d} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Let us trace a cohomology class $[\gamma] \in H^{k-1}_B(M)$ through the Snake Lemma:

1. **Lift:** Pull γ back to the invariant complex via the right inverse of j. From our proof of Proposition 3.5.4, we know the explicit lift is $\alpha = \eta \wedge \gamma$.

2. **Differentiate:** Apply the exterior derivative d to α:

   $$d\alpha = d(\eta \wedge \gamma) = d\eta \wedge \gamma - \eta \wedge d\gamma.$$

   Because $[\gamma]$ is a cohomology class, the representative γ is closed ($d\gamma = 0$). Thus, the second term vanishes, leaving:

   $$d\alpha = d\eta \wedge \gamma \in \Omega^{k+1}_{\text{inv}}(M).$$

3. **Pull back:** We must push this form back to $\Omega^{k+1}_B(M)$ through the inclusion map i. Is $d\eta \wedge \gamma$ basic? Yes. We already know γ is basic, and $d\eta$ is basic because $i_\xi(d\eta) = 0$ (a fundamental Sasakian property) and $\mathcal{L}_\xi(d\eta) = d(\mathcal{L}_\xi \eta) = 0$.

Consequently, the connecting homomorphism operates by taking the wedge product with $d\eta$:

$$\delta([\gamma]) = [d\eta \wedge \gamma]_B = [d\eta]_B \smile [\gamma]_B. \tag{3.5}$$

Exercise 3.5.7. *Fill in all details of the diagram chasing to complete the proof of Theorem 3.5.6. Start by verifying that the map δ is well defined.*

3.6 First Topological Constraints on Sasakian Manifolds

We can now use the Gysin sequence to prove that compact Sasakian manifolds cannot have arbitrary topology. Let us look at the first Betti number $b_1(M) = \dim H^1_{dR}(M)$.

Theorem 3.6.1. *If M is a compact Sasakian manifold, its first Betti number $b_1(M)$ must be **even**.*

Proof. Consider the beginning of the Gysin sequence for $k = 1$:

$$0 \longrightarrow H^1_B(M) \xrightarrow{i^*} H^1_{dR}(M) \xrightarrow{j^*} H^0_B(M) \xrightarrow{[d\eta]\wedge -} H^2_B(M) \longrightarrow \dots \tag{3.6}$$

Note that the sequence maps to 0 prior to H^1_B because $\Omega^{-1}_B(M) = 0$. Assuming M is connected, the only basic 0-forms are constant functions, so $H^0_B(M) \cong \mathbb{R}$. The map out of $H^0_B(M)$ sends $1 \mapsto [d\eta]_B$.

Moreover, by Lemma 3.3.4 we know that $[d\eta]_B$ is not the zero class in $H^2_B(M)$, that is $[d\eta]_B \neq 0$. This means the map $H^0_B(M) \to H^2_B(M)$ given by $c \mapsto c[d\eta]_B$ is **injective**. By exactness at $H^0_B(M)$, the image of $j^* : H^1_{dR}(M) \to H^0_B(M)$ must be the kernel of the injective map, which is $\{0\}$. Thus, j^* is the zero map. By exactness at $H^1_{dR}(M)$, the map $i^* : H^1_B(M) \to H^1_{dR}(M)$ is **surjective**. Since it is also injective (the sequence starts with 0), it is an **isomorphism**.

Thus, $H^1_{dR}(M) \cong H^1_B(M)$. Finally, because M is transversally Kähler, its odd basic Betti numbers are even. Therefore, $b_1(M) = b^B_1(M)$ is even. □

Example 3.6.2. Can the 3-torus T^3 admit a Sasakian structure? $b_1(T^3) = 3$, which is odd. Therefore, T^3 **cannot** admit a Sasakian structure, even though it admits an almost contact metric structure.

3.6.1 Higher Betti Numbers and the Need for a New Theorem

What about higher Betti numbers? Using the Transverse Hard Lefschetz theorem and the Gysin sequence, we can extract partial information about $H^k_{dR}(M)$. A natural question arises: *Is there a Lefschetz-type operator that acts DIRECTLY on $H^*_{dR}(M)$, bypassing the basic cohomology entirely?* In Kähler geometry, $\omega \wedge \cdot$ is an isomorphism on de Rham cohomology. In Sasaki geometry, $d\eta \wedge \cdot$ fails to be an isomorphism on $H^*_{dR}(M)$ (as $[d\eta]_{dR} = 0$ it gives the zero map in de Rham cohomology algebra).

Chapter 4

The Hard Lefschetz Theorem for Sasakian Manifolds

In this chapter, we take the Transverse Hard Lefschetz Theorem as a given. Our main goal is to show how the Gysin sequence algebraically forces a corresponding Hard Lefschetz Theorem on the full de Rham cohomology. We will then use this result to derive stringent topological constraints and definitively separate Sasakian manifolds from strictly K-contact ones.

4.1 The Hard Lefschetz Theorem

The following fundamental theorems were proven in [4].

Theorem 4.1.1 (Hard Lefschetz Theorem for harmonic forms)**.** *Let (M, η, g) be a compact Sasakian manifold of dimension $2n+1$ and $0 \leq k \leq n$. The map*

$$\begin{aligned} \eta \wedge (d\eta)^{n-k} \wedge \cdot : \Omega^k_\Delta(M) &\to \Omega^{2n+1-k}_\Delta(M) \\ \beta &\mapsto \eta \wedge (d\eta)^{n-k} \wedge \beta \end{aligned}$$

*is an **isomorphism**. The inverse is proportional to $\Lambda^{n-k} i_\xi$, where Λ is the adjoint of $L = (d\eta) \wedge \cdot$.*

This isomorphism obviously induces via Hodge theory an isomorphism between the corresponding de Rham cohomology groups.

Theorem 4.1.2 (Hard Lefschetz Theorem in cohomology)**.** *Let (M, η, g) be a compact Sasakian manifold of dimension $2n+1$. For every $0 \leq k \leq n$, the maps*

$$Lef_k : H^k(M) \longrightarrow H^{2n+1-k}(M) \tag{4.1}$$

defined by

$$[\alpha] \longmapsto [\eta \wedge (d\eta)^{n-k} \wedge \Pi_\Delta \alpha], \tag{4.2}$$

are isomorphisms, with $\Pi_\Delta \beta$ denoting the orthogonal projection of β on the space of harmonic forms. The isomorphism Lef_k does not depend on the choice of the Sasakian metric g. It depends only on the contact form η.

Note that, contrary to the symplectic case, we are forced to use the metric structure in the definition of Lef_k. Thus, *a priori*, one could expect that different Sasakian metrics on (M, η) could lead to different Lefschetz isomorphisms. To the utter surprise of the authors, this is not the case. Indeed, you will see from the proof that the Lefschetz isomorphism is independent of the metric. This provides an obstruction for a contact manifold to admit Sasakian structures.

Remark 4.1.3. We will only present the proof of Theorem 4.1.2 here. Establishing the Lefschetz isomorphism directly on harmonic forms requires advanced Hodge theory machinery which we choose to bypass.

4.2 Proof of the Hard Lefschetz Theorem in cohomology. Part 1: the inverse map

4.2.1 Introduction and Setup

Let $(M^{2n+1}, \xi, \eta, \varphi, g)$ be a compact Sasakian manifold of dimension $2n+1$. Recall that we have two primary cohomology theories associated with such a manifold:

1. The standard de Rham cohomology $H^\bullet_{dR}(M)$.
2. The basic cohomology $H^\bullet_B(M, \mathcal{F}_\xi)$ associated with the Reeb foliation, which we will simply denote as $H^\bullet_B$.

These two cohomology theories are linked by the **Gysin long exact sequence**:

$$\cdots \to H^{m-2}_B \xrightarrow{L} H^m_B \xrightarrow{h} H^m_{dR} \xrightarrow{[i_\xi]} H^{m-1}_B \xrightarrow{L} \cdots \tag{4.3}$$

where $L = \cdot \wedge d\eta$ is the Lefschetz operator, h is the natural inclusion map of basic forms into de Rham forms, and $[i_\xi]$ is the interior product with the Reeb vector field.

Because M is a compact Sasakian manifold, we have pointed out that a **basic Hard Lefschetz Theorem** holds. Specifically, for any $j \geq 0$, the map given by repeated wedge product with the fundamental 2-form $d\eta$:

$$L^j : H^{n-j}_B \xrightarrow{\sim} H^{n+j}_B \tag{4.4}$$

is an isomorphism.

Our goal is to fix a degree $k \leq n$ and construct an isomorphism $\mathcal{K} : H^{2n+1-k}_{dR} \to H^k_{dR}$.

4.2.2 Extracting the Short Exact Sequences

Assume $k \leq n$. We will look at the Gysin sequence at two specific degrees to extract two short exact sequences.

4.2.3 The Top Sequence (Degree k)

Consider the segment of the Gysin sequence around degree k:

$$\cdots \longrightarrow H^{k-2}_B \xrightarrow{L} H^k_B \xrightarrow{h} H^k_{dR} \xrightarrow{[i_\xi]} H^{k-1}_B \xrightarrow{L} H^{k+1}_B \longrightarrow \cdots$$

Because $k \leq n$, we are below the middle degree n. By the Basic Hard Lefschetz property:

1. The map $L : H_B^{k-2} \to H_B^k$ is injective, which means $\ker(L) = 0$ (yielding the left side of the short exact sequence).

2. The map $L : H_B^{k-1} \to H_B^{k+1}$ is also injective, meaning $\ker(L) = 0$. By exactness at H_B^{k-1}, this implies $\mathrm{Im}([i_\xi]) = 0$, hence the map $[i_\xi] : H_{dR}^k \to H_B^{k-1}$ is identically zero.

Since $[i_\xi]$ is zero, its kernel is the entire space, $\ker([i_\xi]) = H_{dR}^k$. By exactness at H_{dR}^k, this forces $\mathrm{Im}(h) = \ker([i_\xi]) = H_{dR}^k$, meaning h is surjective. We thus obtain a clean short exact sequence:

$$0 \longrightarrow H_B^{k-2} \xrightarrow{L} H_B^k \xrightarrow{h} H_{dR}^k \longrightarrow 0. \tag{4.5}$$

4.2.4 The Bottom Sequence (Degree $2n + 1 - k$)

Next, we evaluate the Gysin sequence around the de Rham degree $2n + 1 - k$. Shifting the index of the standard sequence, we have:

$$H_B^{2n-k-1} \xrightarrow{L} H_B^{2n-k+1} \xrightarrow{h} H_{dR}^{2n+1-k} \xrightarrow{[i_\xi]} H_B^{2n-k} \xrightarrow{L} H_B^{2n-k+2}$$

Since $k \leq n$, we have $2n - k \geq n$. Above the middle degree, the Lefschetz operator L is surjective. Because $L : H_B^{2n-k-1} \to H_B^{2n-k+1}$ is surjective, exactness dictates that $\mathrm{Im}(h) = 0$, hence $h = 0$. Similarly, $L : H_B^{2n-k} \to H_B^{2n-k+2}$ is surjective. Truncating the sequence yields the second short exact sequence (which we write going right-to-left to match our upcoming diagram):

$$0 \longleftarrow H_B^{2n-(k-2)} \xleftarrow{L} H_B^{2n-k} \xleftarrow{[i_\xi]} H_{dR}^{2n+1-k} \longleftarrow 0 \tag{4.6}$$

4.2.5 The Commutative Diagram

We now align the two short exact sequences vertically. The vertical maps are provided by powers of the Lefschetz operator.

Notice that to go from degree $k-2$ to $2n-k+2$, the degree difference is $2n-2k+4$. Since each application of L increases the degree by 2, we require $n-k+2 = n-(k-2)$ applications. Similarly, to go from degree k to $2n-k$, we require $n-k$ applications of L. By the Basic Hard Lefschetz theorem, both vertical maps are isomorphisms.

We obtain the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_B^{k-2} & \xrightarrow{L} & H_B^k & \xrightarrow{h} & H_{dR}^k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle L^{n-(k-2)}}\,\cong & & \downarrow{\scriptstyle L^{n-k}}\,\cong & & \uparrow{\scriptstyle \mathcal{K}} & & \\
0 & \longleftarrow & H_B^{2n-(k-2)} & \xleftarrow{L} & H_B^{2n-k} & \xleftarrow{[i_\xi]} & H_{dR}^{2n+1-k} & \longleftarrow & 0
\end{array} \tag{4.7}$$

Commutativity of the left square: The left square is comprised entirely of powers of the Lefschetz operator L. Following the arrows from H_B^{k-2} to H_B^{2n-k+2} via the top-right path yields $L \circ L^{n-k} \circ L$, which is exactly L^{n-k+2}. Following the direct left vertical path yields $L^{n-(k-2)} = L^{n-k+2}$. Thus, the left square commutes perfectly.

Because the left square behaves perfectly (it commutes and is made of isomorphisms), the two exact sequences are fundamentally locked together. This suggests defining the natural map $\mathcal{K} : H_{dR}^{2n+1-k} \to H_{dR}^k$ to complete the square on the right:

$$\mathcal{K} := h \circ \left(L^{n-k}\right)^{-1} \circ [i_\xi] \tag{4.8}$$

4.2.6 The Diagram Chase

Theorem 4.2.1. *The induced map $\mathcal{K} : H_{dR}^{2n+1-k} \xrightarrow{\sim} H_{dR}^{k}$ is an isomorphism.*

Proof. We proceed by a standard diagram chase. For brevity, let us define the maps as follows: $f = L$ (top), $g = L$ (bottom), $u = L^{n-(k-2)}$, $v = L^{n-k}$, and $i = [i_\xi]$. The commutativity of the left square implies $g \circ v \circ f = u$.

Part 1: $\mathcal{K}$ is injective. Let $z \in H_{dR}^{2n+1-k}$ such that $\mathcal{K}(z) = 0$. By definition, $h(v^{-1}(i(z))) = 0$. By exactness of the top row ($\ker(h) = \mathrm{Im}(f)$), there exists an element $x \in H_B^{k-2}$ such that

$$f(x) = v^{-1}(i(z)).$$

Applying v to both sides gives $v(f(x)) = i(z)$. Now apply g to both sides to map down to the bottom-left term:

$$g(v(f(x))) = g(i(z)).$$

By exactness of the bottom row, $\mathrm{Im}(i) = \ker(g)$, which means $g(i(z)) = 0$. Therefore,

$$g(v(f(x))) = 0.$$

By the commutativity of the left square, $g \circ v \circ f = u$, so we have $u(x) = 0$. Because u is an isomorphism (by Hard Lefschetz), x must be 0. Consequently, $f(x) = 0$, which implies $v^{-1}(i(z)) = 0$, and thus $i(z) = 0$. Because i is injective (exactness at H_{dR}^{2n+1-k}), we conclude that $z = 0$. Hence, $\mathcal{K}$ is injective.

Part 2: $\mathcal{K}$ is surjective. The surjectivity follows for dimensional reasons, as

$$\dim H_{dR}^{k}(M) = \dim H_{dR}^{2n+1-k}(M)$$

due to Poincaré duality.

Since $\mathcal{K}$ is both injective and surjective, it is an isomorphism. □

4.3 Proof, part 2: Recognizing the direct map

4.3.1 Introduction

Let M^{2n+1} be a compact Sasakian manifold. In our previous construction, we established an isomorphism for $k \leq n$:

$$\mathcal{K} = h \circ \left(L^{n-k}\right)^{-1} \circ [i_\xi] \ : \ H_{dR}^{2n+1-k}(M) \xrightarrow{\sim} H_{dR}^{k}(M) \tag{4.9}$$

where $L = d\eta \wedge \cdot$ is the Lefschetz operator on basic forms, $[i_\xi]$ is the interior product with the Reeb vector field ξ, and h is the map induced in cohomology by the natural inclusion of basic forms into the space of standard differential forms on M.

We now wish to prove that the inverse of this map is given by a simple algebraic wedge product, provided we select the correct representative for the cohomology class.

4.3.2 Tachibana's Theorem

By standard Hodge theory, every de Rham cohomology class in $H^k_{dR}(M)$ contains a unique harmonic representative $\alpha \in \Omega^k(M)$ (with respect to the Sasakian metric g). For $k \leq n$, the geometry of compact Sasakian manifolds places extremely rigid constraints on these harmonic forms.

Theorem 4.3.1 (Tachibana, 1965)**.** *Let M^{2n+1} be a compact Sasakian manifold and let $k \leq n$. If $\alpha \in \Omega^k(M)$ is a harmonic form, then α is:*

1. ***Basic:*** *$i_\xi \alpha = 0$ and $\mathcal{L}_\xi \alpha = 0$.*
2. ***Primitive:*** *$L^{n-k+1}\alpha = (d\eta)^{n-k+1} \wedge \alpha = 0$.*

Because α is harmonic, it is also closed ($d\alpha = 0$). This theorem is the crucial ingredient that ensures our proposed inverse map is both well-defined and perfectly inverts $\mathcal{K}$.

4.3.3 The Proof

Theorem 4.3.2. *Let $k \leq n$ and let $\alpha \in \Omega^k(M)$ be the harmonic representative of a class $[\alpha]_{dR} \in H^k_{dR}(M)$. Define the map $Lef : H^k_{dR}(M) \to H^{2n+1-k}_{dR}(M)$ by:*

$$Lef([\alpha]_{dR}) = [\eta \wedge (d\eta)^{n-k} \wedge \alpha]_{dR} \tag{4.10}$$

Then $Lef = \mathcal{K}^{-1}$.

Proof. Let $\omega = \eta \wedge (d\eta)^{n-k} \wedge \alpha \in \Omega^{2n+1-k}(M)$. We proceed in two steps: first showing that ω is closed (so Lef is well-defined), and second verifying that $\mathcal{K}([\omega]_{dR}) = [\alpha]_{dR}$.

Step 1: ω is a closed form. We compute the exterior derivative of ω using the graded Leibniz rule:

$$\begin{aligned} d\omega &= d\left(\eta \wedge (d\eta)^{n-k} \wedge \alpha\right) \\ &= d\eta \wedge (d\eta)^{n-k} \wedge \alpha - \eta \wedge d\left((d\eta)^{n-k} \wedge \alpha\right) \end{aligned}$$

Since α is basic and closed ($d\alpha = 0$), the term $(d\eta)^{n-k} \wedge \alpha$ is also closed. Thus, $d\left((d\eta)^{n-k} \wedge \alpha\right) = 0$, leaving us with:

$$d\omega = (d\eta)^{n-k+1} \wedge \alpha$$

By Tachibana's Theorem, the harmonic form α is primitive, meaning $(d\eta)^{n-k+1} \wedge \alpha = 0$ strictly at the level of differential forms. Therefore, $d\omega = 0$, and $[\omega]_{dR}$ defines a valid de Rham cohomology class in $H^{2n+1-k}_{dR}(M)$.

Step 2: Lef is the inverse of $\mathcal{K}$. We now apply the operator $\mathcal{K} = h \circ (L^{n-k})^{-1} \circ [i_\xi]$ to the class $[\omega]_{dR}$.

First, we evaluate the interior product i_ξ on the form ω:

$$\begin{aligned} i_\xi \omega &= i_\xi\left(\eta \wedge (d\eta)^{n-k} \wedge \alpha\right) \\ &= (i_\xi \eta) \wedge (d\eta)^{n-k} \wedge \alpha - \eta \wedge i_\xi\left((d\eta)^{n-k} \wedge \alpha\right) \end{aligned}$$

Recall the identities of a Sasakian structure: $i_\xi \eta = 1$ and $i_\xi d\eta = 0$. Furthermore, because α is basic (Tachibana's Theorem), $i_\xi \alpha = 0$. Thus, the second term vanishes entirely:

$$i_\xi \omega = 1 \cdot (d\eta)^{n-k} \wedge \alpha - 0 = (d\eta)^{n-k} \wedge \alpha = L^{n-k}\alpha$$

Since α is basic and closed, $L^{n-k}\alpha$ defines a basic cohomology class. In basic cohomology, this is precisely:

$$[i_\xi \omega]_B = L^{n-k}[\alpha]_B \in H_B^{2n-k}$$

Next, we apply the inverse Lefschetz operator $(L^{n-k})^{-1}$ to this basic class:

$$(L^{n-k})^{-1}\left(L^{n-k}[\alpha]_B\right) = [\alpha]_B \in H_B^k$$

Finally, we apply the natural inclusion map h, which simply maps the basic class back into de Rham cohomology:

$$h([\alpha]_B) = [\alpha]_{dR}$$

Conclusion: We have shown that:

$$\mathcal{K}\left(Lef([\alpha]_{dR})\right) = \mathcal{K}\left([\omega]_{dR}\right) = [\alpha]_{dR}$$

Since $\mathcal{K} \circ Lef = \mathrm{Id}$ on $H^k_{dR}(M)$, and we already established that $\mathcal{K}$ is an isomorphism between finite-dimensional vector spaces, it follows that Lef is precisely the two-sided inverse $\mathcal{K}^{-1}$. □

4.4 Topological Consequences

The Sasaki Hard Lefschetz theorem allows to prove stricter conditions on Betti numbers than the Gysin sequence alone.

Corollary 4.4.1 (Betti Number constraints)**.** *On a compact Sasakian $(2n+1)$-manifold, the Betti numbers satisfy:*

$$b_p(M) \text{ is even for every } \textbf{odd } p \leq n \qquad (\text{and every } \textbf{even } p \geq n+1).$$

Proof. We begin by defining a non-degenerate bilinear form on the de Rham cohomology group $H^p(M)$. Let $x = [\omega], x' = [\omega'] \in H^p(M)$, where ω and ω' are closed p-forms. Let η be the contact 1-form of the Sasakian manifold. We define the bilinear form $B : H^p(M) \times H^p(M) \to \mathbb{R}$ by

$$B(x, x') = \int_M Lef_p(x) \smile x' = \int_M \eta \wedge (d\eta)^{n-p} \wedge \omega \wedge \omega'$$

where $Lef_p : H^p(M) \to H^{2n+1-p}(M)$ given by $Lef_p([\omega]) = [\eta \wedge (d\eta)^{n-p} \wedge \omega]$ is the Hard Lefschetz-type operator for Sasakian manifolds. One can check that it is well defined, e.g. using the fact that each cohomology class has a unique harmonic representative and defining $B(x, x')$ using their harmonic representatives. Or simply observe that if $\omega_1 = \omega + d\alpha$ then the difference in the result is zero as it amounts to the integral of a **basic** $2n+1$-form and all basic forms of degree $2n+1$ are zero.

By commuting the wedge products of the p-forms ω and ω', we introduce a sign change of $(-1)^{p\cdot p} = (-1)^{p^2}$. Thus,

$$\eta \wedge (d\eta)^{n-p} \wedge \omega \wedge \omega' = (-1)^{p^2} \eta \wedge (d\eta)^{n-p} \wedge \omega' \wedge \omega$$

Integrating over M gives $B(x, x') = (-1)^{p^2} B(x', x) = (-1)^p B(x', x)$, since p^2 and p have the same parity.

Next, notice that B is non-degenerate. By the Hard Lefschetz theorem for compact Sasakian manifolds, the operator Lef_p is an isomorphism for $p \leq n$. Furthermore, the standard Poincaré pairing $P : H^{2n+1-p}(M) \times H^p(M) \to \mathbb{R}$, given by $P(y, x') = \int_M y \smile x'$, is non-degenerate. If $B(x, x') = 0$ for all x', then $P(Lef_p(x), x') = 0$ for all x'. By the non-degeneracy of the Poincaré pairing, we must have $Lef_p(x) = 0$. Because Lef_p is an isomorphism, its kernel is trivial, which implies $x = 0$. Hence, B is non-degenerate.

Now, assume $p \leq n$ is **odd**. The sign factor $(-1)^p$ is -1, making B a skew-symmetric bilinear form ($B(x, x') = -B(x', x)$). It is a standard fact from linear algebra that any finite-dimensional vector space admitting a non-degenerate skew-symmetric bilinear form must have an even dimension. Therefore, $b_p(M) = \dim H^p(M)$ must be even.

Finally, we extend this result using Poincaré duality. On a compact orientable $(2n+1)$-manifold, $b_k(M) = b_{2n+1-k}(M)$. Let $k \geq n+1$ be an **even** integer, and set $p = 2n+1-k$. Because $2n+1$ is odd and k is even, their difference p must be odd. Furthermore, since $k \geq n+1$, we have $p \leq 2n+1-(n+1) = n$. Thus, $p \leq n$ is an odd integer. By the first part of the proof, $b_p(M)$ is even, and consequently $b_k(M) = b_p(M)$ is also even. □

4.5 The K-Contact vs. Sasakian Gap

Historically, it was asked whether every K-contact manifold is Sasakian. In dimension 3, this is true. In higher dimensions, we can now use Theorem 4.1.2 to prove it is **false**.

Example 4.5.1. Let us consider the 5-dimensional nilpotent Lie algebra $\mathfrak{g}$ with non-zero Lie brackets

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_1, X_4] = X_5, \quad [X_2, X_3] = X_5. \tag{4.11}$$

Let us denote by α_i the dual 1-form of the vector X_i, for any $i \in \{1, \dots, 5\}$. Then by (4.11) it is easy to compute

$$\begin{aligned} &d\alpha_1 = 0, \quad d\alpha_2 = 0, \quad d\alpha_3 = -\alpha_1 \wedge \alpha_2, \quad d\alpha_4 = -\alpha_1 \wedge \alpha_3, \\ &d\alpha_5 = -\alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3. \end{aligned} \tag{4.12}$$

For example

$$d\alpha_5(X_1, X_4) = -\alpha_5([X_1, X_4]) = -\alpha_5(X_5) = -1, \qquad d\alpha_5(X_2, X_3) = -\alpha_5(X_5) = -1.$$

Here, $d\colon \wedge^k \mathfrak{g}^* \to \wedge^{k+1} \mathfrak{g}^*$ is the Chevalley-Eilenberg differential on the Lie algebra $\mathfrak{g}$. We have $\alpha_5 \wedge (d\alpha_5)^2 = 2\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5$, hence if η is the left-invariant 1-form induced by α_5 on the simply connected Lie group G whose Lie algebra is $\mathfrak{g}$, then η is a contact 1-form on G. The corresponding Reeb vector field is the left-invariant vector field ξ on G induced by X_5.

Now, let us consider a co-compact discrete subgroup Γ of G. Such a subgroup exists by the Malcev criterion, since in our case the structure constants are integers. Then the

quotient $M = G/\Gamma$ is a compact nilmanifold. One can show (see [3] for details) that it admits a K-contact structure. However, one can compute the cohomology and show that:

$$Lef_1 : H^1(M) \longrightarrow H^4(M)$$

is *not* an isomorphism (e.g., it has a non-trivial kernel). Because Theorem 4.1.2 fails, such a manifold **cannot be Sasakian**, even though it is K-contact. Actually, we showed that there is no Sasakian structure on it compatible with the given contact form.

This highlights the "integrability" required by the Sasakian structure (normality) as a major topological constraint, mirroring the gap between almost Kähler and Kähler manifolds.

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